MAT 21C Final Exam

Name:____

Student ID #: ____________________________________

Time and Day of Your Discussion Section#: ________________

Name of Left Neighbor: _____________________________

Name of Right Neighbor: ____________________________

If you are next to the aisle or wall, then please write "aisle" or "wall" appropriately as your left or right neighbor.

- Read each problem carefully.
- **Write every step of your reasoning clearly.**
- Usually, a better strategy is to solve the easiest problem first.
- This is a closed-book exam. You may not use the textbook, crib sheets, notes, or any other outside material. Do not bring your own scratch paper. Do not bring blue books.
- No calculators/laptop computers/cell phones are allowed for the exam. The exam is to test your basic understanding of the material.
- Everyone works on their own exams. Any suspicions of collaboration, copying, or otherwise violating the Student Code of Conduct will be forwarded to the Student Judicial Board.

	Problem #	Score
1	(15 pts)	
2	(10 pts)	
3	(10 pts)	
4	(10 pts)	
5	(10 pts)	
6	(10 pts)	
7	(10 pts)	
8	(15 pts)	
9	(15 pts)	
10	(15 pts)	
11	(15 pts)	
12	(15 pts)	
Total	(150 pts)	

Problem 1 (15 pts) Does the following series converge or diverge? Give reasons for your answer.

(a) (7 pts)

$$\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}.$$

(b) (8 pts)

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}.$$

Hint: You may want to use the fact:

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = \mathrm{e} \approx 2.718.$$

Score of this page:______________

Problem 2 (10 pts) Find the radius and the interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}.$$

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Problem 3 (10 pts) Consider the function $f(x) = \sqrt{1+x}$.

(a) (5 pts) Approximate $f(x)$ by a Taylor polynomial of degree 1 centered at 0. Compute the value of $\sqrt{2}$ using that approximation.

(b) (5 pts) How accurate is this approximation when $0 \leq x \leq 1$? You can express the approximation error either by fraction or by three decimal point number. Then, confirm that the approximate value of $\sqrt{2}$ computed in Part (a) is within this error. Note that the precise value of $\sqrt{2}$ up to 4 digits is 1.414.

Score of this page:______________

Problem 4 (10 pts) Let $P(1, 4, 6)$, $Q(-2, 5, -1)$, $R(1, -1, 1)$.

(a) (5 pts) Find the area of the triangle $\triangle PQR$.

Hint: Length of the cross product of two vectors is equal to the area of a parallelogram formed by those two vectors.

(b) (5 pts) Find the distance from P to the line QR.

Score of this page:______________

Problem 5 (10 pts) Find the limit if it exists, or show that the limit does not exist.

(a) (5 pts)

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}.$$

Hint: Consider $(x, y) \to (0, 0)$ along the line $y = x$ and along the parabola $y = x^2$.

(b) (5 pts)

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}.$$

Hint: Consider the limit in the polar coordinates (r, θ).

Score of this page:______________

Problem 6 (10 pts) Verify that the function of two variables

$$u(x,t) = \mathrm{e}^{-\alpha^2 k^2 t} \sin kx$$

is a solution of the heat conduction equation:

$$u_t = \alpha^2 u_{xx}.$$

Score of this page:______________

Problem 7 (10 pts) Assuming that the equation

$$x\mathrm{e}^{y} + \sin(xy) + y - \ln 2 = 0$$

defines y as a differentiable function of x, use the Implicit Differentiation Theorem to find the value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at the point $(x, y) = (0, \ln 2)$.

Score of this page:______________

Problem 8 (15 pts) Find the direction in which $f(x,y) = \sin x + e^{xy}$

(a) (5 pts) Increases most rapidly at the point $(0,1)$,

(b) (5 pts) Decreases most rapidly at the point $(0,1)$.

(c) (5 pts) Does not change (i.e., is flat) at the point $(0,1)$.

Score of this page:______________

Problem 9 (15 pts) Consider the sphere with radius $r > 0$ in 3D, $x^2 + y^2 + z^2 = r^2$.

(a) (7 pts) Find the tangent plane at the point $\left(\frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}\right)$ on this sphere.

(b) (8 pts) Show that *every* normal line to this sphere passes through the center of the sphere, i.e., the origin.

Hint: Pick any point (a, b, c) on this sphere, and consider the normal line at that point.

Score of this page:______________

Problem 10 (15 pts) Let $f(x, y) = 2x^2 + y^2$.

(a) (8 pts) Find the linearization at the point $(1, 1)$. Then use it to approximate $f(1.1, 0.9)$. Compare the approximate value with the true value.

(b) (7 pts) Approximate $f(2, 2)$ using the same linearization. Compare the approximate value with the true value. At which point is the linear approximation better, $(1.1, 0.9)$ or $(2, 2)$?

Score of this page:______________

Problem 11 (15 pts) Consider the following function over the closed domain $D = \{(x,y) \mid -\pi \leq x \leq \pi, -\pi \leq y \leq \pi\}$:

$$f(x,y) = x\cos y.$$

(a) (7 pts) Find the local maxima, local minima, saddle points of f if any.

(b) (8 pts) Find the absolute maxima and absolute minima.

Score of this page:______________

Problem 12 (15 pts) Use Lagrange multipliers to find the maximum and minimum values of the function

$$f(x, y) = \mathrm{e}^{xy} \quad \text{subject to} \quad x^2 + y^2 = 1.$$

Note that we only consider the real values for x and y, not the complex values.

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