Name:____________________________

Student ID #:____________________________

Final

MAT-022A

You have 100 minutes. You may only use a pencil (or pen) and the scrap paper that I provide. No calculators, notes or books. You must show your work to receive full credit.

1. Let $\mathbf{u}$ and $\mathbf{v}$ be orthogonal vectors such that $||\mathbf{u}|| = 2$ and $||\mathbf{v}|| = 1$. Find $||2\mathbf{u} - 3\mathbf{v}||$. (Your work must show your answer is true for any $\mathbf{u}$ and $\mathbf{v}$ that satisfy the above properties.) (10 points)

2. Find an orthonormal basis for the subspace of $\mathbb{R}^3$ spanned by $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$. (10 points)

3. Let N be the set of all 3×3 nonsingular matrices with real entries. Prove or disprove that N is a real vector space under the operations of matrix addition and scalar multiplication. (10 points)

4. Let P_n denote the set of all polynomials of degree at most n with real coefficients (i.e. polynomials of the form $a_0+a_1t+a_2t^2+...+a_nt^n$ for $a_0, a_1, ..., a_n \in \mathbb{R}$) and let $S = \{t^2, 1+t, t^3+4\}$. (15 points)

a) Is the set S linearly independent in P_3?

b) Is the set S a basis for P_3?

c) What is $dim(P_n)$? (no work necessary)

5. Let $A = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 1 & 2 & 4 \end{bmatrix}$. (15 points)

a) Find a basis for the null space of A.

b) Find a basis for the row space of A.

c) Find a basis for the column space of A.

d) What is the nullity of A?

e) What is the rank of A?

6. If $nullity(A) = 2$ for some 6×8 matrix A, what is the rank of A? (5 points)

7. $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ and $T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ are ordered bases for the set of all 2×2 diagonal matrices with real entries). Find the transition matrix $P_{S \leftarrow T}$ from the T-basis to the S-basis. (10 points)

8. Let $P_{S \leftarrow T} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$ be a transition matrix from some ordered basis T to another ordered basis S of the same vector space. Find $P_{T \leftarrow S}$, the transition matrix from the S-basis to the T-basis. (10 points)

9. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. Find an orthogonal matrix Q such that $Q^T A Q = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (15 points)

Extra Credit 1: Prove that for an $n \times n$ matrix A, the matrix transformation $f(\mathbf{x}) = A\mathbf{x}$ is both one-to-one (i.e. $f(\mathbf{x}) = f(\mathbf{x}) \Leftrightarrow \mathbf{x} = \mathbf{y}$) and onto (i.e. for each $\mathbf{y}$ in $\mathbb{C}^n$ there is an $\mathbf{x}$ in $\mathbb{C}^n$ such that $f(\mathbf{x}) = \mathbf{y}$) if and only if A is nonsingular. (10 points, test score can't exceed 100 points)

Extra Credit 2: Let A be a 3×3 matrix that rotates vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \epsilon \mathbb{R}^3$ around the x-axis by 45 degrees. Find a basis for the subspace of $\mathbb{R}^3$ spanned by the eigenvectors of A. (Hint: You don't need to find A. Just ask yourself, "What is special about eigenvectors?") (10 points, test score can't exceed 100 points)