Mathematics 22B, Section 2
The Final Examination, December 9, 2015

Instructions: **Work all problems in your bluebook.** Only the bluebook will be collected.

Notation: $\mathbb{R}$ = field of real numbers, ODE=ordinary differential equation, $\mathrm{i} = \sqrt{-1}$, the constant e is the unique positive real number satisfying $\int_1^{\mathrm{e}} \frac{dt}{t} = 1$; that is, the natural logarithm of e equals 1.

#1. (10 pts) Consider the matrix ODE

$$M\frac{d^2u}{dt^2} + Au = 0 \tag{1}$$

where $u \in \mathbb{R}^N$, M is a $N \times N$ diagonal matrix with positive entries, and A is a $N \times N$ real symmetric matrix with positive eigenvalues. Both M and A are independent of t. If we assume a solution of the form

$$u(t) = \mathrm{e}^{\mathrm{i}\omega t} f$$

where $f \in \mathbb{R}^N$, $f \neq 0$, is independent of t, show that ω^2 must satisfy the equation

$$\det\left(A - \omega^2 M\right) = 0.$$

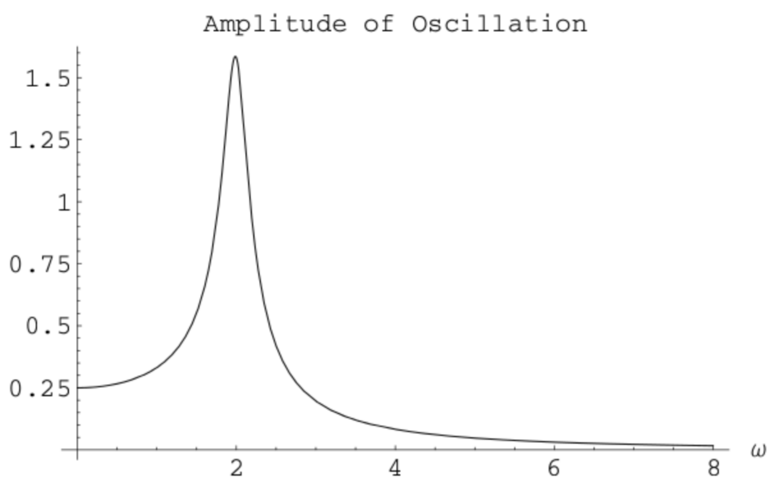

Figure 1: Figure for question #2. Amplitude as a function of driving frequency ω.

#2. (10 pts) A simple (classical) harmonic oscillator (i.e. a mass-spring system), initially at rest, is subjected to an external driving force $F(t) = \cos \omega t$. The resulting amplitude of oscillation of the oscillator, as a function of the driving frequency ω, is displayed in Figure 1. Assuming that the frictional forces are very small; and hence negligible, on the basis of this graph *estimate the period* of the oscillator. Give a reason for your answer.

#3. (20 pts) The two-dimensional Laplace equation in polar coordinates ($x = r\cos\theta$ and $y = r\sin\theta$) is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{2}$$

Assume a solution of the form (separation of variables)

$$u(r,\theta) = R(r)\Theta(\theta).$$

1. Find ODEs for $R(r)$ and $\Theta(\theta)$.
2. Solve the Θ ODE. (You are not asked to solve the R ODE.)
3. What can you say about the separation constant, i.e. does it have to take on special values; and if so, why?

#4. (20 pts) Consider the nonlinear, coupled ODEs for the unknowns $f_1(x)$, $f_2(x)$ and $f_3(x)$, $x \in \mathbb{R}$:

$$\begin{aligned}\frac{df_1}{dx} &= f_2(x)\, f_3(x),\\ \frac{df_2}{dx} &= -f_1(x)\, f_3(x),\\ \frac{df_3}{dx} &= -k^2 f_1(x)\, f_2(x),\end{aligned}$$

with initial conditions $f_1(0) = 0$, $f_2(0) = 1$, $f_3(0) = 1$ and k is a constant, $0 \le k \le 1$. Show that

$$f_1(x)^2 + f_2(x)^2 = 1 \quad \text{and} \quad k^2 f_1(x)^2 + f_3(x)^2 = 1$$

for all x.

#5. (20 pts) Consider the 2nd order (scalar) ODE

$$\frac{d^2 y}{dx^2} - x\,y = 0, \quad -\infty < x < \infty. \tag{3}$$

Note that this is *not* a constant coefficient ODE.

1. We assume a power series solution of (3) of the form

$$y(x) = y_0 + y_1 x + y_2 x^2 + y_3 x^3 + \cdots = \sum_{n=0}^{\infty} y_n x^n \tag{4}$$

where the coefficients y_n are to be determined. Substitute (4) into (3) and show that

$$y_2 = 0 \tag{5}$$

and

$$(n+2)(n+1)y_{n+2} - y_{n-1} = 0 \quad \text{for} \quad n = 1, 2, 3, \ldots. \tag{6}$$

Show that it follows from (5) and (6) that

1.
$$0 = y_2 = y_5 = y_8 = y_{11} = \cdots = y_{3n+2} = \cdots$$

2.
$$y_{3n} = y_0 \prod_{j=0}^{n-1} \frac{1}{(3j+3)(3j+2)}, \quad n = 1, 2, 3, \ldots$$

3.
$$y_{3n+1} = y_1 \prod_{j=0}^{n-1} \frac{1}{(3j+4)(3j+3)}, \quad n = 1, 2, 3, \ldots.$$

#6. (20 pts) In this problem you may assume as given

1.
$$\int_{-\infty}^{\infty} \mathrm{e}^{-ax^2+2bx}\, dx = \sqrt{\frac{\pi}{a}}\, \mathrm{e}^{b^2/a}, \quad a > 0.$$

2. If $f(x)$ is continuous and rapidly decreasing at infinity, the *Fourier transform* of f is
$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-2\pi \mathrm{i}\, x\xi}\, dx$$
and the *inverse Fourier transform* is
$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i}\, x\xi}\, d\xi.$$

Consider now the one-dimensional heat equation on the line $(-\infty < x < \infty)$
$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$$
with initial condition
$$u(x, 0) = f(x), \;\; -\infty < x < \infty.$$
Show that the solution $u(x, t)$ is given by
$$u(x, t) = \int_{-\infty}^{\infty} K(x, y; t) f(y)\, dy$$
where
$$K(x, y; t) = \frac{1}{\sqrt{4\pi t}}\, \mathrm{e}^{-\frac{(x-y)^2}{4t}}.$$

END OF EXAM