Mathematics 22B, Section 1
The Final Examination, March 20, 2015

Instructions: **Work all problems in your Bluebook.** Only the bluebook will be collected.

Notation: $\mathbb{R}$ = field of real numbers, ODE=ordinary differential equation.

#1. (75 points) Consider the matrix ODE

$$\frac{d^2u}{dt^2} + Au = 0 \tag{1}$$

where $u \in \mathbb{R}^N$ and A is a $N \times N$ real symmetric matrix with constant coefficients. We assume A has N distinct, *positive* eigenvalues λ_n with corresponding eigenvectors $f_n \in \mathbb{R}^N$; that is,

$$Af_n = \lambda_n f_n, \;\; n = 1, 2, \ldots N.$$

Recall the linear algebra fact that the vectors f_n are orthogonal and form a basis for $\mathbb{R}^N$. You may assume they are *orthonormal*; that is

$$(f_n, f_m) = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

where $(\cdot, \cdot)$ is the standard inner (dot) product on $\mathbb{R}^N$.

1. Given the initial conditions

$$u(0) = u_0 \;\text{ and }\; \frac{du}{dt}(0) = v_0$$

where u_0 and v_0 are given vectors in $\mathbb{R}^N$, show that the solution to (1) satisfying these initial conditions is

$$u(t) = \sum_{n=1}^{N} \left[(u_0, f_n) \cos(\omega_n t) f_n + \frac{1}{\omega_n} (v_0, f_n) \sin(\omega_n t) f_n \right]$$

where $\omega_n = \sqrt{\lambda_n}$. Note: (u_0, f_n) is the inner (dot) product of u_0 and f_n, and similarly for (v_0, f_n).

2. Now consider the *inhomogeneous* ODE

$$\frac{d^2u}{dt^2} + Au = \cos(\omega t) f \tag{2}$$

where f is a fixed constant coefficient vector in $\mathbb{R}^N$ and ω is a positive real number. We assume $\omega \neq \omega_n$ for all n.

(a) Assume a *particular* solution of (2) of the form

$$u_p(t) = \cos(\omega t) g$$

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where g is a constant coefficient vector in $\mathbb{R}^N$. Show that if we expand f in the basis vectors f_n; that is,

$$f = \sum_{n=1}^{N} \alpha_n f_n, \quad \alpha_n \in \mathbb{R},$$

then

$$g = \sum_{n=1}^{N} \frac{\alpha_n}{\omega_n^2 - \omega^2} f_n$$

(b) Given the initial conditions

$$u(0) = 0 \ \text{ and } \ \frac{du}{dt}(0) = 0,$$

show that the solution to (2) satisfying these initial conditions is

$$u(t) = \sum_{n=1}^{N} \left[\frac{\alpha_n}{\omega_n^2 - \omega^2} \left(\cos(\omega t) - \cos(\omega_n t) \right) \right] f_n$$

#2. (75 points) Consider the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

In class we solved this equation on the real line $\mathbb{R}$, the half-line $\mathbb{R}^+$ and the circle S. In this problem we solve the heat equation on the line segment $[0, L]$ subject to the boundary conditions

$$u(0,t) = 0 \ \text{ and } \ u(L,t) = 0 \ \text{ for all } \ t > 0 \tag{3}$$

with initial condition

$$u(x,0) = f(x) \ \text{ for } \ 0 < x < L. \tag{4}$$

We assume $f(x)$ is a continuous function with $f(0) = f(L) = 0$.

1. If we assume a solution of the form (separate variables)

$$u(x,t) = X(x)T(t),$$

find ODEs that $X(x)$ and $T(t)$ must satisfy. Solve these ODEs. (The arithmetic is a bit easier if you call the separation constant $-k^2$.)

2. Using the solutions found in part (1), apply the boundary conditions (3) and find the allowed values of the constant k. This should give you a sequence of solutions $u_n(x,t) = X_n(x)T_n(t)$.
3. Show that the solution $u(x,t)$ satisfying the boundary conditions (3) and the initial conditions (4) can be written as

$$u(x,t) = \int_0^L K(x,y,t) f(y)\, dy$$

(continued on next page)

where

$$K(x,y,t) = \frac{2}{L}\sum_{n=1}^{\infty} \sin(\frac{n\pi}{L}x)\ \sin(\frac{n\pi}{L}y)\ \mathrm{e}^{-(\pi^2 n^2/L^2)t}$$

Hint: The integral

$$\frac{2}{L}\int_0^L \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)\,dx = \begin{cases} 1 & \text{if} \ \ m=n \\ 0 & \text{if} \ \ m\neq n \end{cases}$$

will be helpful (m and n are integers).

#3. (50 points) Consider the 2nd order (scalar) ODE

$$\frac{d^2y}{dx^2} - x\,y = 0, \quad -\infty < x < \infty. \tag{5}$$

Note that this is *not* a constant coefficient ODE.

1. We assume a power series solution of (5) of the form

$$y(x) = y_0 + y_1 x + y_2 x^2 + y_3 x^3 + \cdots = \sum_{n=0}^{\infty} y_n x^n \tag{6}$$

where the coefficients y_n are to be determined. Substitute (6) into (5) and show that

$$y_2 = 0 \tag{7}$$

and

$$(n+2)(n+1)y_{n+2} - y_{n-1} = 0 \ \text{ for } \ n = 1,2,3,\ldots. \tag{8}$$

2. Show that it follows from (7) and (8) that

(a)

$$0 = y_2 = y_5 = y_8 = y_{11} = \cdots = y_{3n+2} = \cdots$$

(b)

$$y_{3n} = y_0 \prod_{j=0}^{n-1} \frac{1}{(3j+3)(3j+2)}, \quad n = 1,2,3,\ldots$$

(c)

$$y_{3n+1} = y_1 \prod_{j=0}^{n-1} \frac{1}{(3j+4)(3j+3)}, \quad n = 1,2,3,\ldots.$$