Name: ______________________________

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Signature: ______________________________

Final Exam Math 22B-2 Winter 2008

Instructions:

- Fill in the top of this page when you receive your exam but **do not** begin until told to do so.
- There are 6 questions. The exam is out of 155 points.
- Read each question and take each step carefully. Arithmetic errors can make the problem more difficult and will cost you time.
- Show work for credit. Write up your work **clearly**– remember that you are trying to communicate your answer to your reader.

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1. ____________________/17

2. ____________________/23

3. ____________________/27

4. ____________________/25

5. ____________________/35

6. ____________________/28

Total: ____________________/155

Table of elementary Laplace Transform

$f(t) = \mathcal{L}^{-1}\{F(x)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, \;\; s > 0$
e^{at}	$\frac{1}{s-a}, \;\; s > a$
t^n (n positive integer)	$\frac{n!}{s^{n+1}}, \;\; s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}, \;\; s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}, \;\; s > 0$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \;\; s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \;\; s > a$
$t^n e^{at}$ (n positive integer)	$\frac{n!}{(s-a)^{n+1}}, \;\; s > a$
$f^n(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{n-1}(0)$
$(-t)^n f(t)$	$F^n(s)$

1. (17pts) Solve the initial valued problem

$$y' = \frac{2x}{x^2y + y}, \quad y(0) = -3,$$

expressing your solution in explicit form.

2. (23pts) A tank initially contains 120 liters of pure water. A mixture containing a concentration of 80 g/liter of salt enters the tank at a rate of 2 liters/min, and the well-stirred mixture leaves the tank at the same rate.

   (a) Let $Q(t)$ be the amount of salt in the tank at any time, where t is in minutes. Set up the initial valued problem to model the variations in the amount of salt in the tank.

   (b) Find the amount of salt in the tank at the end of 10 min.

   (c) Find the limiting amount of salt in the tank as $t \to \infty$.

3. (27pts) The motion of a mass on a spring can be described by the solution of the initial value problem

$$mu'' + \gamma u' + ku = F(t), \quad u(0) = u_0, \quad u'(0) = u_0'.$$

A mass weighing 8 lb stretches a spring 6 in. The mass is attached to a viscous damper with a damping constant of 2 lb-sec/ft., and it is acted on by an external force of $\cos 3t$ lb. The mass is displaced 2 in. downward and released.

(a) Formulate the initial valued problem describing the motion of the mass.

(b) Solve the initial valued problem in part (a) using either the Method of Undetermined Coefficients or Variation of Parameters.

4. (25pts) (a) Let $f(t)$ be a piecewise continuous function of exponential order as $t \to \infty$. State the definition of the Laplace Transform $F(s)$ of f.

(b) Use the Laplace Transform (no credit for any other method) to solve the initial valued problem

$$y'' + 2y' + 5y = 0, \quad y(0) = 5,\ y'(0) = -4.$$

5. (35pts) Find the general solution of the nonhomogeneous system of differential equations

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t.$$

6. (28pts) Consider the third-order homogeneous differential equation with constant coefficients

$$\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - 4\frac{dy}{dt} - 8y = 0.$$

(a) Transform it into a system of first-order differential equations $\mathbf{x}' = A\mathbf{x}$.

(b) The coefficient matrix A has eigenvalues $r_1 = 2,\ r_2 = r_3 = -2$, and two linearly independent eigenvectors

$$\xi_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \quad \text{and} \quad \xi_2 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

corresponding to eigenvalues 2 and -2, respectively. These two eigenvectors form two linearly independent solutions of the system. To find the third solution, we assume that it is of the form $\mathbf{x}(t) = \xi t e^{-2t} + \eta e^{-2t}$.

What conditions must ξ and η satisfy? Prove it.

(c) Find ξ and η.

(d) What is the general solution of the system? (be sure to remove any redundancy from the solution).

(e) What is the general solution of the original third-order differential equation?