MAT 25B Final Exam (2013/12/12)

Honor Pledge: I pledge on my honor that I have not given or received any unauthorized assistance on this exam.

Name: Signature:

1. Show all your work. Jumping to right answers without minimum reasoning deserves no credit.
2. Answer each numbered problem on a separate answer sheet.
3. Open book. No electronic devices are allowed.
4. Read directions to each problem carefully.
5. Neatness and organization are also important.

Problem #	Points	
1	40	
2	20	
3	40	
4	30	
5	30	
6	20	
7	20	
Total Points	200	

 # 1. [40 pts] Determine whether the following statements are true or false. Provide counterexamples for those that are false, and supply proofs for those that are true. (10 points each.)

(1) If $a < 1$ for all $a \in A$, then $\sup A < 1$.

(2) The union of an arbitrary collection of compact sets is compact.

(3) Let a be a real number. Then, the set $\{a\}$ is closed.

(4) If A and B are connected, then $A \cup B$ is connected.

 # 2. [20 pts] Let (a_n) be a sequence such that $a_n \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = a$. Show that

$$\lim_{n\to\infty} \sqrt{a_n} = \sqrt{a}.$$

3. [40 pts] Determine whether the following series converges or diverges. (10 points each.)

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n}3^n}{10^n}$ (2) $\displaystyle\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$ (3) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+7}{n^2+5n}$ (4) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$

 # 4. [10 points each.] Let (a_n) be a sequence defined as

$$a_1 = 1, \quad a_{n+1} = \frac{4a_n + 5}{5} \quad \text{for } n \in \mathbb{N}.$$

(1) Show that the sequence (a_n) satisfies $a_n < 5$ for all $n \in \mathbb{N}$.

(2) Show that the sequence (a_n) is increasing.

(3) Show that the sequence (a_n) converges. Compute $\lim_{n\to\infty} a_n$.

 # 5. Let A and B be subsets in $\mathbb{R}$.

(1) [10 pts] Show that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$

(2) [20 pts] Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

 # 6. [20 pts] Let (a_n) be a sequence such that $\lim_{n\to\infty} a_n = a$. Show that the following set is compact

$$K = \{a_n : \ n \in \mathbb{N}\} \cup \{a\}.$$

 # 7. [20 pts] Let A and B be connected sets in $\mathbb{R}$ such that $A \cap B \neq \emptyset$. Show that $A \cup B$ is connected.

Do I contradict myself?
Very well then I contradict myself.
(I am large, I contain multitudes.)
– Walt Whitman