University of California Davis **Name (Print):** ______________________

Intro. to Abs. Math. MAT 108 **Student ID (Print):** ______________________

Final Examination **December 10 2018**

Time Limit: 120 Minutes

This examination document contains 8 pages, including this cover page, and 7 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	15	
6	15	
7	10	
Total:	100	

1. (15 points) Prove the following two statements.

   (a) (8 points) Show that for every $k, n \in \mathbb{N}$, $k \leq n$, we have the inequality

   $$\binom{n}{k} \leq 2^n.$$

   If it helps, you can use the equality $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

   (b) (7 points) Prove that for all $n \in \mathbb{N}$, the integer number

   $$2^{2n} + 5 \cdot 3^{4n-1} - 5^{6n}$$

   is divisible by $6 \in \mathbb{N}$.

2. (15 points) Show that the following sequences (x_n) converge to the indicated limit by using the ε-definition of the limit.

   (a) (8 points) $\lim_{n\to\infty} \frac{n!}{n^n+1} = 0.$

   (b) (7 points) $\lim_{n\to\infty} \frac{3n^2+2n+1}{4n^2+7} = \frac{3}{4}.$

3. (15 points) Let (x_n), $n \in \mathbb{N}$, be a sequence of real numbers that satisfies the recursion

$$x_{n+1} = \frac{2x_n}{1 + x_n},$$

with the initial value $x_1 = 2$.

(a) (8 points) Show that the sequence is decreasing and bounded below.

(b) (7 points) Show that the sequence is convergent and find its limit.
If it helps, you can use that $x_n \geq 0.5$ for all $n \in \mathbb{N}$.

4. (15 points) Consider the set

$$X = \{\sqrt{q} : q \in \mathbb{Q} \text{ and } 0 \leq \sqrt{q} < e\}.$$

(a) (8 points) Show that $\inf(X) = 0$ and $\sup(X) = e$.

(b) (7 points) Show that X admits an injection into the power set $P(\mathbb{N})$. Does it admit a surjection onto $P(\mathbb{N})$?

5. (15 points) Prove the following two statements.

   (a) (8 points) Show that $\sqrt[4]{17}$ is **not** a rational number.

   (b) (7 points) Give an example of a sequence (x_n) of rational numbers which converges to an irrational number $L \in \mathbb{R}$. Is it possible for a sequence (y_n) of irrational numbers to converge to a rational number ?

6. (15 points) Consider the function

$$f : \mathbb{N} \longrightarrow \mathbb{R}^+, \quad f(x) = \sqrt{x^2 + 25}.$$

(a) (8 points) Show that f is **not** a surjection, but it is an injection.

(b) (7 points) Show that the function

$$g : \mathbb{R} \longrightarrow [5, \infty), \quad g(x) = \sqrt{x^2 + 25},$$

is a surjection, but it is **not** an injection.

7. (10 points) **Prove** or **disprove** the following assertions

(a) (5 points) Every positive real number $r \in \mathbb{R}^+$ can be written in the form

$$r = \sqrt{q}$$

for a rational number $q \in \mathbb{Q}$.

(b) (5 points) Let X, Y be arbitrary sets of the same cardinality[1], and $f : X \longrightarrow Y$ a surjection. Then f is necessarily a bijection.

[1] Recall that, by definition, the equality of cardinalities $\mathrm{card}(X) = \mathrm{card}(Y)$ is equivalent to the existence of a bijection between the two sets X and Y.