Partial Differential Equations

Math 118A, Fall 2013

Final

NAME..

I.D. NUMBER...

Closed book. Explain your answers fully. Unless stated otherwise, you use standard results from lectures or the text provided you state them clearly.

Question	Points	Score
1	20	
2	30	
3	30	
4	30	
5	25	
6	25	
7	40	
Total	200	

1. [20 pts] For each of the following PDEs for $u(x, y)$, give their order and say if they are nonlinear or linear. If they are linear, say if they are homogeneous or nonhomogeneous and if they have constant or variable coefficients.

(a) $u_x = (\sin x)u_y$

(b) $uu_x + u_y = u_{xx} + \sin x$

(c) $u_{xxyy} = \sin x$

2. [30 pts] Solve the following initial value problem for $u(x,t)$:

$$u_t + 3u_x = \sin t, \qquad u(x,0) = \sin x.$$

3. [30 pts] (a) Solve the following initial-boundary value problem for the heat equation for $u(x,t)$:

$$\begin{aligned} &u_t = u_{xx} && 0 < x < 1, \quad t > 0, \\ &u_x(0,t) = 0, \quad && u_x(1,t) = 0, \qquad t > 0, \\ &u(x,0) = f(x) && 0 \leq x \leq 1. \end{aligned}$$

(b) What type of boundary conditions are these? How does your solution behave as $t \to +\infty$? Give a physical explanation of this behavior.

4. [30 pts] Use separation of variables to solve the following Dirichlet problem for Lapace's equation in polar coordinates for $u(r, \theta)$ in the unit disc $r < 1$:

$$\begin{aligned}
&\frac{1}{r}\left(r u_r\right)_r + \frac{1}{r^2} u_{\theta\theta} = 0, \\
&u(1, \theta) = 1 \qquad\quad \text{if } 0 < \theta < \pi, \\
&u(1, \theta) = -1 \qquad \text{if } \pi < \theta < 2\pi.
\end{aligned}$$

5. [25 pts] (a) For all (smooth) functions $X(x)$, $Y(x)$, prove that

$$\int_a^b (XY'' - YX'')\, dx = [XY' - YX']_a^b\,.$$

(b) Suppose that $X_1(x)$, $X_2(x)$ are solutions of the eigenvalue problem

$$\begin{aligned} -X_1'' &= \lambda_1 X_1, \qquad & X_1(a) &= 3X_1(b), \quad & 3X_1'(a) &= X_1'(b), \\ -X_2'' &= \lambda_2 X_2, \qquad & X_2(a) &= 3X_2(b), \quad & 3X_2'(a) &= X_2'(b), \end{aligned}$$

where $\lambda_1 \neq \lambda_2$ are distinct, real eigenvalues. Show that X_1 and X_2 are orthogonal, meaning that $\int_a^b X_1 X_2\, dx = 0$.

6. [25 pts] Let Ω be a bounded open set in $\mathbb{R}^2$ with boundary $\partial\Omega$.

(a) Suppose that $u(x, y)$ is a solution of the PDE

$$u_{xx} + u_{yy} - u = 0.$$

Show that u cannot attain a maximum value at any point of Ω where $u > 0$, or a minimum value at any point of Ω where $u < 0$.

(b) Let $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ be given functions. Show that a solution of the following Dirichlet boundary value problem is unique:

$$\begin{aligned} &u_{xx} + u_{yy} - u = f \qquad \text{in } \Omega, \\ &u = g \qquad \text{on } \partial\Omega, \end{aligned}$$

7. [40 pts] Let c, V be positive constants, and consider the PDE

$$u_{tt} + 2V u_{xt} + \left(V^2 - c^2\right) u_{xx} = 0.$$

(a) Show that the change of variables

$$u(x,t) = w(\xi, \tau), \qquad \xi = x - Vt, \quad \tau = t$$

transforms the PDE into the wave equation $w_{\tau\tau} - c^2 w_{\xi\xi} = 0$.
(b) Solve the initial value problem

$$\begin{aligned}
&u_{tt} + 2V u_{xt} + \left(V^2 - c^2\right) u_{xx} = 0, \qquad -\infty < x < \infty, \quad t > 0,\\
&u(x,0) = \phi(x), \qquad -\infty < x < \infty,\\
&u_t(x,0) = \psi(x), \qquad -\infty < x < \infty.
\end{aligned}$$

(c) Describe the domains of dependence and influence for this PDE and sketch them in the (x,t)-plane. Consider the cases: (i) $0 < V < c$; (ii) $0 < c < V$.