119A Final Examination

Question 1:

(i) Sketch the function

$$v(x) = \alpha + (x-1)^2(x+1)^2$$

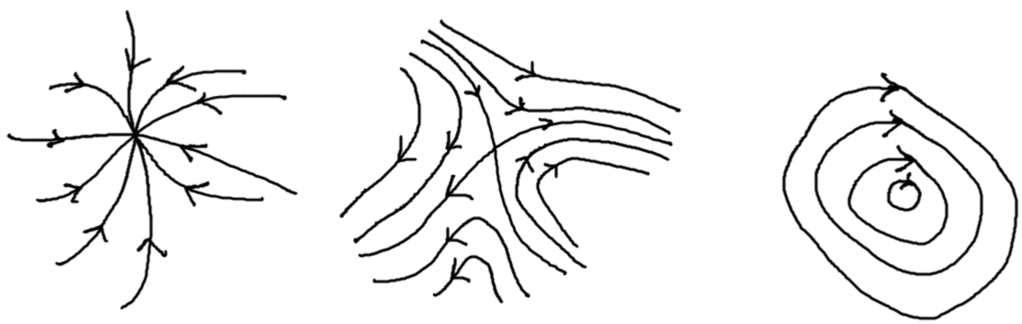
for various choices of α. Make sure you include at least one choice for each possible cardinality of the solution set $\{x \,|\, v(x) = 0\}$.

(ii) Use your sketches to draw a rough x vs. α bifurcation diagram for the system

$$\dot{x} = v(x)\,.$$

(iii) Use your bifurcation diagram to describe in words and/or pictures the behavior of this system at the choices of α you made above.

Question 2: Trajectories solving the system $\dot{\vec{x}} = \vec{v}$ for three different vector fields $\vec{v}$ are drawn below:



(i) Sketch vector fields $\vec{v}$ that could lead to each of the three sets of tractories.

(ii) Draw (in a different color if available) a loop around the fixed point on each of your three vector fields.

(iii) Try to follow the direction of the vector field as you move once clockwise around each loop you have drawn. How many times does the vector rotate as you go around the loop? (It might help to redraw your loops marking various values of the vector field along the loop.)

(iv) Your answers to the previous question should be an integer (where negative integers are used to count anticlockwise rotations). This called the *index* of the loop. We are interested to hear any thoughts you have on how the index could be used to classify the fixed points contained inside a given loop.

Question 3: Athos, Porthos, and Aramis are preparing to run a marathon by running around a circular track. To amp up their training regimen, they decide to organize their Thursday session as follows: All three runners will begin running in the same direction from the same place at exactly the same time. They will maintain their (somewhat differing) personal maximum speeds until such time when their positions on the track once again are all in the same place (which need not be their starting position). At that time the training session ends.

(i) According to your intuition, for how long will Athos, Porthos and Aramis train?

(ii) Develop a 2D, first order dynamical system to describe the training system. Sketch some phase space trajectories and analyze the system's evolution. Was your intuition correct? Discuss.

Question 4:

Consider the system

$$\dot{x} = r \log x + x - 1 \,.$$

(i) Show the system has a fixed point $x_* = 1$.

(ii) Linearize the system about x_*. What happens when $r = -1$?

(iii) To handle study the system better when $r \approx -1$, extend your linearization result in part (ii) to include terms of order $(x - x_*)^2$.

(iv) Your approximation in part (iii) can be further simplified by using the variable $X = (x - x_*)/\alpha$. Find α such that the system becomes

$$\dot{X} \approx (r+1)X - X^2 \,.$$

(v) Now draw an X versus r bifurcation diagram and classify the bifurcation you find.

Question 5:

Consider the system

$$\varepsilon \ddot{x} + \dot{x} + \alpha - x^2 = 0 \,.$$

(i) Suppose $0 < \varepsilon \approx 0$. Explain how you could analyze the system in this case and discuss all possible qualitatively different behaviors as α varies.

(ii) Now rewrite the second order equation as an equivalent pair of first order equations of the form $\dot{\vec{x}} = \vec{v}$. Sketch the vector field $\vec{v}$.

(iii) How many initial conditions does the 2D, first order dynamical system in part (ii) require to uniquely determine a solution? How many initial conditions did your method in part (i) require? Use the vector field you sketched in part (ii) to reconcile these two answers.

Question 6:

Consider the matrix

$$M = \begin{pmatrix} 0 & \lambda \\ \lambda & \alpha \end{pmatrix}.$$

(i) For which values of α and λ does the matrix provide a basis of eigenvectors?

(ii) Analyze all possible distinct behaviors of the 2D, first order dynamical system

$$\dot{\vec{x}} = M\vec{x}\,.$$

Include a sketch of the type of trajectories you expect in the xy-plane.