Math 125A
Final Exam

Printed Name ______________________________
(FIRST) (LAST)

Student ID number ______________________________

Please Show All Your Work, and Mark Your Answers Clearly.

No Calculators -- No Scratch Paper -- No Cell Phones

There are **8 pages** of problems. (The last page is for extra credit.)

You are expected to do your own work, and to adhere to the UCD Code of Academic Conduct.

Please write your proofs as clearly and completely as possible.

Please indicate clearly if you continue work on the back of a page.

Have a good break!

① LET $f(x) = \sin \frac{1}{\sqrt[3]{x}}$ FOR $x \neq 0$, AND $f(0) = 0$.

SHOW THAT f IS DISCONTINUOUS AT 0.

10 PTS

② LET $f_n \to f$ UNIFORMLY ON D. PROVE THAT IF EACH f_n IS CONTINUOUS AT $c \in D$, THEN f IS CONTINUOUS AT c.

15 PTS

③ LET f BE CONTINUOUS ON AN INTERVAL I AND DIFFERENTIABLE ON I^{o}. IF $f'(x) \geq 0$ FOR ALL $x \in I^{o}$, SHOW THAT f IS INCREASING ON I.

10 PTS

④ SHOW THAT $f(x) = \sum_{n=1}^{\infty} \frac{n^4 \cos(n x^8)}{x^2 + 5^n}$ IS A CONTINUOUS FUNCTION ON $\mathbb{R}$.

16 PTS

(3) GIVE AN EXAMPLE OF A FUNCTION $f: \mathbb{R} \to \mathbb{R}$ WITH THE FOLLOWING PROPERTY:
(NO JUSTIFICATION REQUIRED)

A) f IS DISCONTINUOUS AT EVERY $x \in \mathbb{R}$.

6
PTS

B) f IS CONTINUOUS ONLY AT THE IRRATIONAL NUMBERS.

9
PTS

C) f IS INFINITELY DIFFERENTIABLE, AND $f(x) > 0$ FOR $x \in (-4, 5)$
AND $f(x) = 0$ FOR $x \notin (-4, 5)$.

9
PTS

⑥ IF $f_n(x) = \dfrac{x^2 + 5x^n}{x^n + 1}$ FOR $x \geq 0$,

A) FIND $f(x) = \lim_{n \to \infty} f_n(x)$.

9 PTS

B) SHOW THAT $f_n \to f$ UNIFORMLY ON $[3, \infty)$.

11 PTS

⑦ LET $f(x) = e^{-\frac{1}{x^4}}$ IF $x \neq 0$, AND $f(0) = 0$,

FIND $f'(0)$ USING THE DEFINITION OF THE DERIVATIVE, OR SHOW THAT IT DOES NOT EXIST.

14 PTS

⑧ USE LAGRANGE'S FORM OF THE REMAINDER TO SHOW THAT

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$ FOR ALL X (GIVEN THAT THIS IS THE MACLAURIN SERIES FOR $f(x) = e^x$).

13 PTS

⑨ PROVE THE FOLLOWING CASE OF L'HOSPITAL'S RULE:

LET f AND g BE DIFFERENTIABLE ON (a,b) WITH $g'(x) \neq 0$ ON (a,b).

IF $\lim_{x \to a^+} f(x) = 0$ AND $\lim_{x \to a^+} g(x) = 0$, AND IF $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$, THEN $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$.

15 PTS

(10) LET $f: S \to T$ WHERE (S, d) AND (T, d') ARE METRIC SPACES.

A) PROVE THAT f IS CONTINUOUS ON S IFF FOR EVERY OPEN SET U IN T, $f^{-1}(U)$ IS OPEN IN S.

16 PTS

B) IF f IS CONTINUOUS AND E IS A COMPACT SUBSET OF S, PROVE THAT $f(E)$ IS COMPACT IN T.

14 PTS

(11) PROVE THAT IF f IS CONTINUOUS ON $[a,b]$, THEN f IS UNIFORMLY CONTINUOUS ON $[a,b]$.

18
PTS

(12) PROVE THAT IF $e \le b < c$, THEN $b^c > c^b$.

15
PTS

(13) PROVE THE FOLLOWING STATEMENT OF THE CHAIN RULE:
IF g IS DIFFERENTIABLE AT c AND f IS DIFFERENTIABLE AT $g(c)$,
THEN $f \circ g$ IS DIFFERENTIABLE AT c AND $(f \circ g)'(c) = f'(g(c))\,g'(c)$.

15
PTS

(EXTRA
CREDIT)