Name: ______________________

Final

Wait! Do not turn this page until told to.

No books, notes, phones, or calculators.

Show all of your work.

Justify every statement that you make.

Each of the 4 problems is worth 30 points.

Any points above 100 are a bonus.

Good luck!

1	2	3	4	

for office use

1. Let X and Y be two discrete random variables.

   (a) Define the *correlation* of X and Y.

   (b) Prove that $V(X+Y) - V(X) - V(Y) = 2\text{Cov}(X,Y)$. Justify every step.

   (c) Let X be the number of ⚅, and let Y be the total number of ⚀, ⚂ and ⚄, that occur in n rolls of a fair die. What is the probability mass function of $X+Y$?

   (d) Compute the correlation of X and Y as in part (c).

2. Let X be a normal random variable with mean μ and variance σ^2.

   (a) Prove that $Z = \frac{X-\mu}{\sigma}$ is a standard normal random variable.

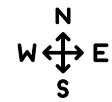
   (b) Find the density of the *log-normal random variable* $Y = e^X$.

   (c) Evaluate the following limits, where $\varepsilon > 0$.

   - $\lim_{\varepsilon \to 0} P(\mu - \varepsilon < X \leq \mu + \varepsilon) =$

   - $\lim_{\varepsilon \to 0} \dfrac{P(\mu - \varepsilon < X \leq \mu + \varepsilon)}{2\varepsilon} =$

3. An ant takes a random step (X_1, Y_1) in the Euclidean plane $\mathbb{R}^2$, in one of the four cardinal directions: east, north, west, or south, according to the following distribution.



step	$(1,0)$	$(0,1)$	$(-1,0)$	$(0,-1)$
$P((X_1,Y_1) = \text{step})$	0.4	0.3	0.2	0.1

(a) Compute $E(X_1)$, $E(Y_1)$, $V(X_1)$, and $V(Y_1)$.

(b) Starting out at $(0, 0)$, the ant takes a sequence of 100 independent steps. Each step starts where the previous one ends, having the same distribution as above. Let (X, Y) be its final position. Find $E(X)$, $E(Y)$, $V(X)$, and $V(Y)$.

(c) Reminder: The *Euclidean distance* between the points (a, b) and (c, d) is $\sqrt{(a-c)^2 + (b-d)^2}$. Let D be the Euclidean distance between (X, Y) and $(E(X), E(Y))$. Find $E(D^2)$.

4. Consider a sequence of independent trials, each of which is a success with probability $\frac{1}{2}$. Let:

$X =$ the number of failures preceding the first success

$Y =$ the number of failures between the first two successes

$Z =$ the number of failures between the second and third successes

Find the following conditional probabilities, for any non-negative integers a, b.

(i) $P(X = a \mid X \geq b)$

(ii) $P(X = a \mid X + Y = b)$

(iii) $P(X = a \text{ and } Y = b \mid X + Y + Z = 5)$

Selected Distributions

Distribution	Probability Mass/Density	Support	Expectation
$\mathrm{B}(n,p)$	$p_X(k) \;=\; \binom{n}{k} p^k (1-p)^{n-k}$	$0, 1, \ldots, n$	np
$\mathrm{G}(p)$	$p_X(k) \;=\; (1-p)^{k-1} p$	$1, 2, 3, \ldots$	$\frac{1}{p}$
$\mathrm{Po}(\lambda)$	$p_X(k) \;=\; e^{-\lambda}\, \lambda^k / k!$	$0, 1, 2, \ldots$	λ
$\mathrm{U}(a,b)$	$f_X(t) \;=\; 1/(b-a)$	$t \in [a,b]$	$\frac{a+b}{2}$
$\mathrm{Exp}(\lambda)$	$f_X(t) \;=\; \lambda e^{-\lambda t}$	$t \in [0,\infty)$	$\frac{1}{\lambda}$
$\mathrm{N}(\mu,\sigma^2)$	$f_X(t) \;=\; \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(t-\mu)^2/2\sigma^2}$	$t \in \mathbb{R}$	μ

Useful Limits

$$\left(1 + \frac{x}{m}\right)^m \;\xrightarrow[m\to\infty]{}\; e^x$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \;=\; e^x$$

$$n! \;\sim\; \frac{n^n}{e^n}\sqrt{2\pi n}$$