Name: ______________________

# Final

**Wait!** Do not turn this page until told to.

No books, notes, phones, or calculators.

Show all of your work.

Justify every statement that you make.

Each of the 6 problems is worth 20 points.

Any points above 100 are a bonus.

Good luck!

| 1 | 2 | 3 | 4 | 5 | 6 | |
|---|---|---|---|---|---|---|
| | | | | | | |

for office use

1. Suppose that $A$, $B$ and $C$ randomly and independently choose 3, 4, and 5 objects respectively, out of 10 objects. Find the expected number of objects

   (a) chosen by all three – by $A$, $B$ and $C$?

   (b) chosen by at least one of $A$, $B$ and $C$?

2. Let $X$ be an exponential random variable with rate 1. Find the density of the following.

(a) $Y = -\log X$

(b) $Z = e^{-X}$

3. A monkey types an infinite sequence of independent random characters, each one is equally likely to be A, B, C, D, ..., Z. Let $X$ be the number of characters typed until both U and C appear at least once.

   (a) Find the probability that $X$ is greater than 52, and simplify it using $(1 - \frac{1}{n})^n \approx \frac{1}{e}$.

   (b) Find the the expected value and the variance of $X$.
   Hint: Represent $X$ as a sum of two numbers.

4. Jill's bowling scores are approximately normally distributed with mean 170 and standard deviation 20, while Jack's scores are approximately normally distributed with mean 160 and standard deviation 15. Jack and Jill each bowl $n$ games. Assume that all scores are independent. Let $A_n$ be the event that Jack's total score is higher.

   (a) $P(A_1) =$

   (b) $P(A_4) =$

   (c) $\lim_{n \to \infty} P(A_n) =$

5. (a) Give an example for two uncorrelated random variables $X, Y$ that are not independent.

(b) Two ambulances travel back and forth along a one mile road between $A$ and $B$. Assume that their locations are uniformly distributed on the road and independent. Let $X$ and $Y$ be their distances from the point $A$, such that $X$ is the distance of the ambulance that is closer to $A$, and $Y$ is the distance of the farther one.

Explain briefly why the joint pdf of $X$ and $Y$ is given by $f(x,y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$

What is the correlation between $X$ and $Y$?

6. Let $A$ be the event that a die would land on its ⚅ face 45 times or more in 180 independent rolls. Denote by $q_0$ the probability of $A$ given that the die is fair.

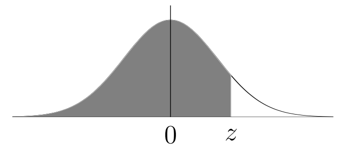
   (a) Recall Chebyshev's inequality: $P\left(|X - E(X)| \geq k\,\mathrm{sd}(X)\right) \leq \frac{1}{k^2}$ for $k > 0$
   What does it tell you about $q_0$?

   (b) Find an approximation of $q_0$.

   (c) Is it reasonable to assume that the die is biased if $A$ has occurred? How would your answer depend on the following two parameters?
   $r =$ the percentage of biased dice in the population;
   $q_1 =$ the probability of $A$ given a biased die.

# Cumulative Normal Distribution Table

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-s^2/2}\, ds$$




| $z$ | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| 0.1 | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 |
| 0.2 | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| 0.3 | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | .6480 | .6517 |
| 0.4 | .6554 | .6591 | .6628 | .6664 | .6700 | .6736 | .6772 | .6808 | .6844 | .6879 |
| 0.5 | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 |
| 0.6 | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 |
| 0.7 | .7580 | .7611 | .7642 | .7673 | .7704 | .7734 | .7764 | .7794 | .7823 | .7852 |
| 0.8 | .7881 | .7910 | .7939 | .7967 | .7995 | .8023 | .8051 | .8078 | .8106 | .8133 |
| 0.9 | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | .8315 | .8340 | .8365 | .8389 |
| 1.0 | .8413 | .8438 | .8461 | .8485 | .8508 | .8531 | .8554 | .8577 | .8599 | .8621 |
| 1.1 | .8643 | .8665 | .8686 | .8708 | .8729 | .8749 | .8770 | .8790 | .8810 | .8830 |
| 1.2 | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | .8962 | .8980 | .8997 | .9015 |
| 1.3 | .9032 | .9049 | .9066 | .9082 | .9099 | .9115 | .9131 | .9147 | .9162 | .9177 |
| 1.4 | .9192 | .9207 | .9222 | .9236 | .9251 | .9265 | .9279 | .9292 | .9306 | .9319 |
| 1.5 | .9332 | .9345 | .9357 | .9370 | .9382 | .9394 | .9406 | .9418 | .9429 | .9441 |
| 1.6 | .9452 | .9463 | .9474 | .9484 | .9495 | .9505 | .9515 | .9525 | .9535 | .9545 |
| 1.7 | .9554 | .9564 | .9573 | .9582 | .9591 | .9599 | .9608 | .9616 | .9625 | .9633 |
| 1.8 | .9641 | .9649 | .9656 | .9664 | .9671 | .9678 | .9686 | .9693 | .9699 | .9706 |
| 1.9 | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |
| 2.0 | .9772 | .9778 | .9783 | .9788 | .9793 | .9798 | .9803 | .9808 | .9812 | .9817 |
| 2.1 | .9821 | .9826 | .9830 | .9834 | .9838 | .9842 | .9846 | .9850 | .9854 | .9857 |
| 2.2 | .9861 | .9864 | .9868 | .9871 | .9875 | .9878 | .9881 | .9884 | .9887 | .9890 |
| 2.3 | .9893 | .9896 | .9898 | .9901 | .9904 | .9906 | .9909 | .9911 | .9913 | .9916 |
| 2.4 | .9918 | .9920 | .9922 | .9925 | .9927 | .9929 | .9931 | .9932 | .9934 | .9936 |
| 2.5 | .9938 | .9940 | .9941 | .9943 | .9945 | .9946 | .9948 | .9949 | .9951 | .9952 |
| 2.6 | .9953 | .9955 | .9956 | .9957 | .9959 | .9960 | .9961 | .9962 | .9963 | .9964 |
| 2.7 | .9965 | .9966 | .9967 | .9968 | .9969 | .9970 | .9971 | .9972 | .9973 | .9974 |
| 2.8 | .9974 | .9975 | .9976 | .9977 | .9977 | .9978 | .9979 | .9979 | .9980 | .9981 |
| 2.9 | .9981 | .9982 | .9982 | .9983 | .9984 | .9984 | .9985 | .9985 | .9986 | .9986 |
| 3.0 | .9987 | .9987 | .9987 | .9988 | .9988 | .9989 | .9989 | .9989 | .9990 | .9990 |
| 3.1 | .9990 | .9991 | .9991 | .9991 | .9992 | .9992 | .9992 | .9992 | .9993 | .9993 |
| 3.2 | .9993 | .9993 | .9994 | .9994 | .9994 | .9994 | .9994 | .9995 | .9995 | .9995 |
| 3.3 | .9995 | .9995 | .9995 | .9996 | .9996 | .9996 | .9996 | .9996 | .9996 | .9997 |
| 3.4 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9998 |

# Selected Distributions

| Distribution | Probability Mass/Density | Support | E | V |
|---|---|---|---|---|
| $\mathrm{B}(n,p)$ | $p_X(k) \;=\; \binom{n}{k} p^k (1-p)^{n-k}$ | $0, 1, \ldots, n$ | $np$ | $np(1-p)$ |
| $\mathrm{G}(p)$ | $p_X(k) \;=\; (1-p)^{k-1} p$ | $1, 2, 3, \ldots$ | $\frac{1}{p}$ | $\frac{1}{p^2} - \frac{1}{p}$ |
| $\mathrm{Po}(\lambda)$ | $p_X(k) \;=\; e^{-\lambda}\, \lambda^k / k!$ | $0, 1, 2, \ldots$ | $\lambda$ | $\lambda$ |
| $\mathrm{U}(a,b)$ | $f_X(t) \;=\; 1/(b-a)$ | $t \in [a, b]$ | $\frac{a+b}{2}$ | $\frac{(b-a)^2}{12}$ |
| $\mathrm{Exp}(\lambda)$ | $f_X(t) \;=\; \lambda e^{-\lambda t}$ | $t \in [0, \infty)$ | $\frac{1}{\lambda}$ | $\frac{1}{\lambda^2}$ |
| $\mathrm{N}(\mu, \sigma^2)$ | $f_X(t) \;=\; \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(t-\mu)^2/2\sigma^2}$ | $t \in \mathbb{R}$ | $\mu$ | $\sigma^2$ |