

Name: \_\_\_\_\_

# Final

**Wait!** Do not turn this page until told to.

No books, notes, phones, or calculators.

Show all of your work.

Justify every statement that you make.

Each of the 4 problems is worth 30 points.

The maximum is 100 points.

Good luck!

1	2	3	4	
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for office use

1. Let  $c_n$  the number of ways to arrange  $n$  distinct cards in a row of piles.

*Note: The number of piles and their sizes can be any positive numbers. The order of the piles in the row matters, as well as the order of cards in each pile. Examples:  $c_1 = 1$ ,  $c_2 = 4$ .*

(a) Find, by any method, the exponential generating function of  $c_n$ .

(b) Find, by any method, a simple formula for  $c_n$ .

2. (a) Define the Stirling number of the second kind  $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ , and the Bell number  $b(n)$ .
- (b) Prove the inequality  $b(n) \leq n^n$ .
- (c) Prove the inequality  $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} \geq k^{n-k}$  for every  $k \leq n$ .

3. A country uses as currency coins with values of \$1, \$2, and \$5.

(a) Let  $f_{n,k}$  = the number of different ways to pay \$ $n$  with  $k$  coins.


*Examples:*  $f_{3,1} = 0$ ,  $f_{3,2} = 1$ ,  $f_{3,3} = 1$ ,  $f_{8,4} = 2$




Write in a simple form the generating function  $F(x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} f_{n,k} x^n y^k$

(b) Let  $g_n$  = the number of ways to pay \$ $n$  with any number of coins.

*Examples:*  $g_0 = 1$ ,  $g_1 = 1$ ,  $g_2 = 2$ ,  $g_3 = 2$ ,  $g_4 = 3$ ,  $g_5 = 4$

Find  $G(x) = \sum_{n=0}^{\infty} g_n x^n$ , and a recurrence formula for  $g_n$  given  $g_{n-1}, g_{n-2}, \dots, g_{n-8}$ .

4. We paint a set of six mailboxes arranged like this:   
Two colorings are equivalent if one is obtained from the other by reflections and rotations.

*Example:*   $\sim$    $\sim$  

- (a) How many black/white colorings are there, up to this equivalence?
- (b) What is the cycle indicator for the action of these symmetries on the set of mailboxes?
- (c) How many non-equivalent colorings have 2 yellow, 2 red, and 2 blue mailboxes?