Math 150b: Modern Algebra
Final Exam

Please defend your answers in complete sentences. Since this is an exam, short justifications are good enough, but try to say things that hold water.

All problems are worth the same amount.

1. How many lines through the origin are there in the $\mathbb{F}_5$-vector space $\mathbb{F}_5^3$? How many lines are there that may or may not pass through the origin? (The latter are defined as cosets of lines through the origin.)

2. Let V be the quotient vector space $\mathbb{R}[\{a,b,c,d\}]/\langle a+2b+3c+4d\rangle$. Find a basis for V, find a non-zero dual vector $\phi : V \to \mathbb{R}$, and find its values on the vectors a, b, c, and d.

3. The real vector space $V = \mathbb{R}^3$ is also a vector space over the rational numbers $\mathbb{Q}$, if we forget part of scalar multiplication. Give an example of three vectors in V that are linearly independent over $\mathbb{Q}$, but linearly dependent over $\mathbb{R}$.

4. The quotient ring $\mathbb{Q}[x]/(x^3+2x^2+4x+2)$ is a field because the polynomial in the denominator is irreducible. Find the reciprocal $\frac{1}{x+1}$ in this ring.

5. Draw a diagram of the ideal $I = (1+i\sqrt{5}, 1-i\sqrt{5})$ in the ring $R = \mathbb{Z}[\sqrt{-5}]$, and describe the quotient ring R/I. (It is isomorphic to a much more standard ring which you should find.)

6. Let $R = \mathbb{R}[[x]]$ be the ring of formal power series over the real numbers. If a or $a(x)$ is an element of R, let $h(a)$ be the degree of the first non-zero term of a. Is h a Euclidean height for this ring?

7. Let $R = M_3(\mathbb{Z})$ be the ring of 3×3 matrices over the integers. Does R have a non-zero nilpotent element? Does it have a zero divisor which is not nilpotent?

8. Suppose that an abelian group A has presentation matrix

$$M = \begin{pmatrix} 97 & 137 \\ 73 & 103 \end{pmatrix}.$$

This is pretty complicated, so I will give you the hint that the determinant of this matrix is -10. Prove that just from that hint, there is only one possibility for the Smith normal form of M. What, then, is the isomorphism type of A?