One-dimensional range-restricted C^2 interpolation algorithm

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A Problem...

Given n points in the x-y plane, how can we pass a graph through them with second derivatives "as small as possible"?

And what if we require the graph to be above the x-axis or between two horizontal lines?

Three-points-interpolation

$$f_1 = (x-2)^2 + 2 \tag{1}$$

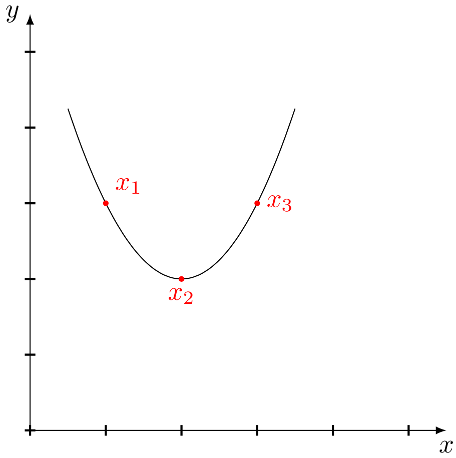


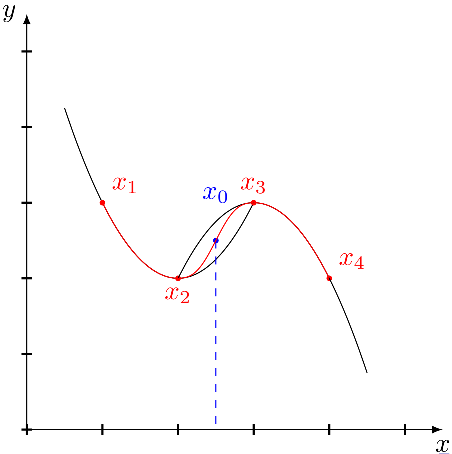
Figure: use quadratic function to minimize the second derivative

Patching

$$f_1 = (x-2)^2 + 2 \tag{2}$$

$$f_2 = -(x-3)^2 + 3 \tag{3}$$

$$g(x) = \theta(x) f_1(x) + (1-\theta(x)) f_2(x) \tag{4}$$



When $\theta(x)$ is linear

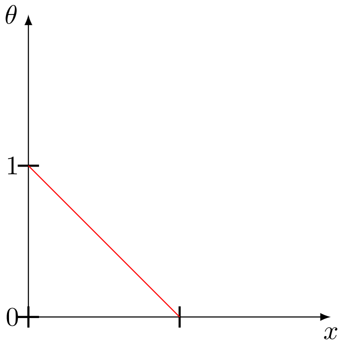


Figure: linear

When $\theta(x)$ is cubic

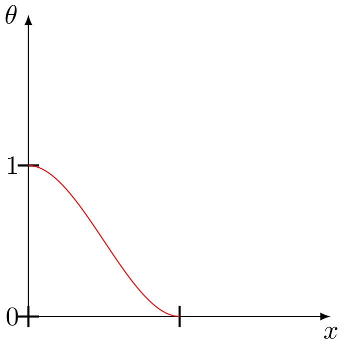


Figure: cubic

When $\theta(x)$ is quintic

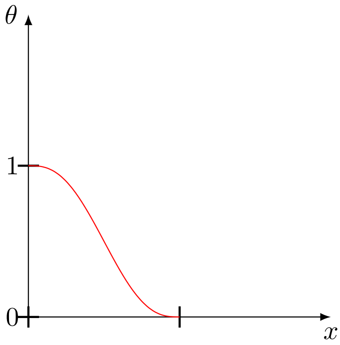


Figure: quintic

When $\theta(x)$ is a bump function

$$\theta(x) = \begin{cases} e \times \exp(-\frac{1}{1-x^2}) & x \in (-1,1) \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

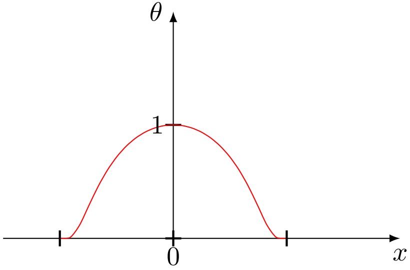


Figure: bump function

Interpolation without lower bound

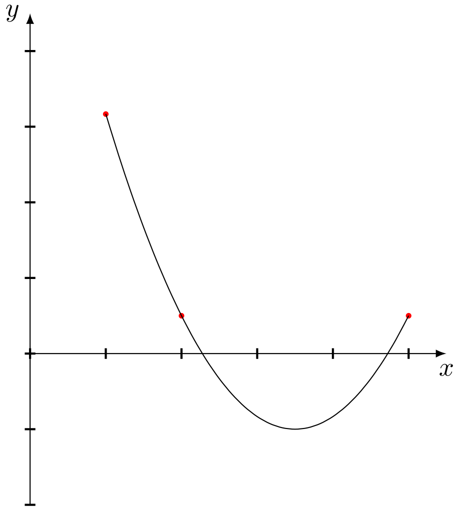


Figure: We cannot guarantee that quadratic is non-negative

One-jet interpolation

One-jet is the tangent line at that point.

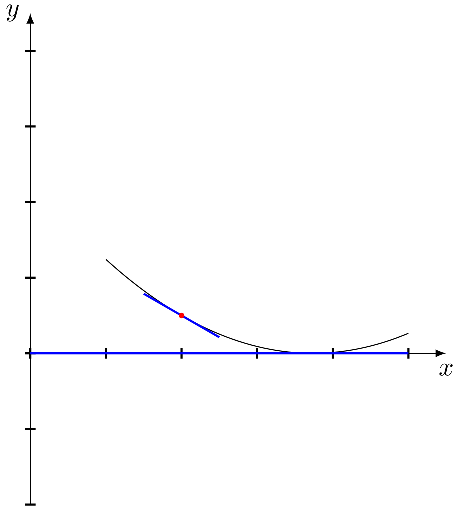


Figure: One-jet interpolation

Three points interpolation with lower bound

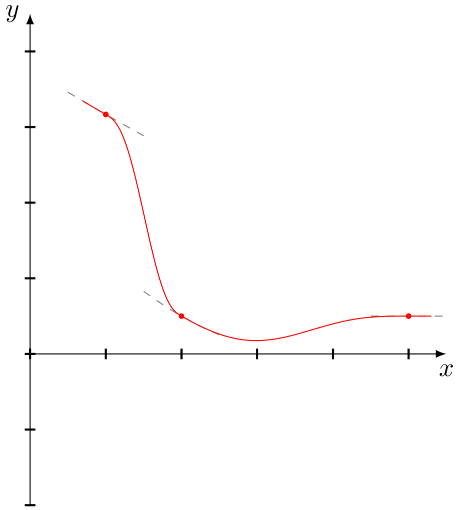


Figure: Three points interpolation that satisfies the lower bound condition

Finding optimal slopes

Whitney norm

$$\sum_{j=1,2,3}\sum_{m=0,1}\left|\frac{d^m}{dx^m}p_j(x_j)\right| + \sum_{\substack{i,j=1,2,3\\ i\neq j}}\sum_{m=0,1}\frac{\left|\frac{d^m}{dx^m}(p_i-p_j)(x_i)\right|}{|x_i-x_j|^{2-m}}$$

$$= \sum_{j=1,2,3}(|b_j|+|k_j|) + \sum_{\substack{i,j=1,2,3\\ i\neq j}}\left(|b_i-b_j+k_j\delta_{ij}|\delta_{ij}^{-2}+|k_i-k_j|\delta_{ij}\right) \tag{6}$$

where

$$p_j(x) = k_j(x-x_j)+b_j$$

and

$$\delta_{ij} = |x_i-x_j|$$

Optimization problem

$$L_S = \begin{bmatrix} \delta_{21}^{-2} & 0 & -\delta_{21}^{-2} & \delta_{21}^{-1} & 0 & 0 \\ 0 & 0 & \delta_{32}^{-2} & 0 & -\delta_{32}^{-2} & \delta_{32}^{-1} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \delta_{21}^{-1} & 0 & -\delta_{21}^{-1} & 0 & 0 \\ 0 & 0 & 0 & \delta_{32}^{-1} & 0 & -\delta_{32}^{-1} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (7)$$

$$\beta = L_S \begin{pmatrix} b_1 \\ k_1 \\ b_2 \\ k_2 \\ b_3 \\ k_3 \end{pmatrix} \quad (8)$$

Optimization problem

$$L_\varphi = \operatorname{diag}\left(0, \frac{1}{2\sqrt{y_1}}, 0, \frac{1}{2\sqrt{y_2}}, 0, \frac{1}{2\sqrt{y_3}}\right) \tag{9}$$

The problem becomes minimizing

$$\|L_\varphi \beta\|_{\ell_2}^2 + \|\beta\|_{\ell_1} \tag{10}$$

subject to

$$\operatorname{diag}(1,0,1,0,1,0)\, L_S^{-1}\beta = \begin{pmatrix} y_1 \\ 0 \\ y_2 \\ 0 \\ y_3 \\ 0 \end{pmatrix} \tag{11}$$

Karush-Kuhn-Tucker (KKT) method

Optimize $f(\vec{x})$ subject to

$$g_i(\vec{x}) \geq 0, h_j(\vec{x}) = 0. \quad (12)$$

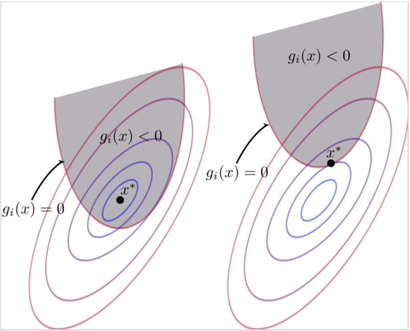


Figure: Visualization of KKT method (Source: Wikipedia, image by Onmyphd at http://www.onmyphd.com/?p=kkt.karush.kuhn.tucker)

Two boundaries

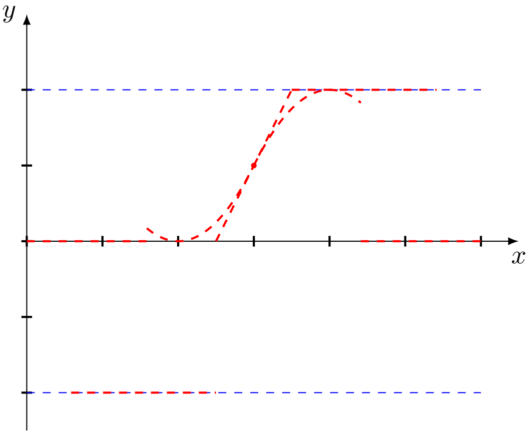


Figure: The one jet interpolation with two boundaries (before patching)

Two boundaries

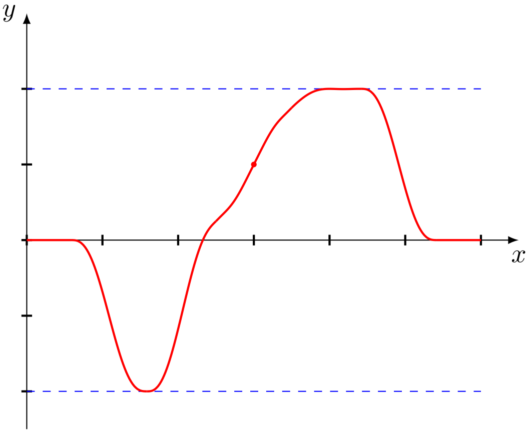


Figure: The one jet interpolation with two boundaries

Demo and questions