ALGEBRA PRELIM EXAM SEPTEMBER 2022

(1) Let M, N be normal solvable subgroups of a group G. Show that their product MN is also solvable.

(2) (a) Let H be a Sylow p-subgroup of a finite group G and let K be a subgroup of G. Is it always true that $H \cap K$ is a Sylow p-subgroup of K? Justify your answer.

(b) Prove that there are no simple groups of order 312, 616, or 1960. (*Note:* $312 = 2^3 \cdot 3 \cdot 13$, $616 = 2^3 \cdot 7 \cdot 11$, $1960 = 2^3 \cdot 5 \cdot 7^2$)

(3) Let F/k be a (finite) Galois extension, let $k \subseteq K \subseteq L$, and let L/K be a (finite) Galois extension. Suppose L and F are both contained in a larger field. Prove that $L \cap F/K \cap F$ is Galois.

(4) How many idempotents are there in the ring

$$R = \mathbb{Q}(\omega) \otimes_{\mathbb{Q}} \mathbb{Q}[x]/(x^4 - 16)$$

where $\omega \in \mathbb{C}$ is a primitive 3rd root of unity? Explain.

(5) Find all prime ideals in the ring $\mathbb{Q}[x]/(x^4 + 4x^2)$.

(6) (a) Let $R = \mathbb{C}[x] \supseteq I = (x)$. Is I a free (left) R-module? Why or why not?

(b) Let $S = \mathbb{C}[x, y] \supseteq J = (x, y)$. Is J a free (left) S-module? Why or why not?