TOPOLOGY PRELIM EXAM SEPTEMBER 2022

You can use a major structural theorem from the course/syllabus without proof if you state it clearly and explain exactly how you are applying it in your solution. In specific examples, you **must** give a **complete proof** for computations of invariants, existence or properties of maps, homeomorphisms, and homotopies, and existence or properties of specific spaces, with the following two exceptions. You can use without proof the values of $\pi_1(S^n)$ for all n and $\pi_n(S^n)$ for all n.
Note that S^n denotes the n-dimensional sphere and T^2 denotes the 2-torus $S^1 \times S^1$.

(1) Let T be the 3-dimensional solid tetrahedron, and T_k its k-skeleton. In other words, T_1 is the union of vertices and edges of T, T_2 is the union of vertices, edges, and faces, and $T_3 = T$.
  (a) Find the fundamental group of T_1.
  (b) Find the fundamental group of T_2.
  (c) Find the fundamental group of T_3.

(2) For each of the following pairs of spaces X and Y and values n, does there exist a covering map $p : X \to Y$ of degree n? If yes, give an explicit construction of the covering map. IF no, prove it does not exist.
  (a) $X = T^2$, $Y = S^2$, $n = 2$.
  (b) $X = T^2$, $Y = T^2$, $n = 3$.
  (c) $X = S^2$, $Y = T^2$, $n = \infty$.

(3) Let $B \subset \mathbb{R}^3$ be the unit ball

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$$

and ∂B its 2-sphere boundary.
  (a) Show that B is contractible, i.e. construct a homotopy from the identity to a constant map.
  (b) Consider the restriction of the map $a : \mathbb{R}^3 \to \mathbb{R}^3$ given by $a(x, y, z) = (-x, -y, -z)$ to the boundary ∂B of B. Let X be the quotient space $B/\sim$ where $p \sim q$ iff $p, q \in \partial B$ and $a(p) = q$. Let $x_0 = [(0, 0, 1)]$ be the equivalence class of $(0, 0, 1)$ in X. Calculate $\pi_1(X, x_0)$.
  (c) Consider the loop in X which is the image of the segment $\{(0, 0, t) \mid -1 \leq t \leq 1\}$ in the quotient space X. What is the order of this loop in $\pi_1(X, x_0)$?

(4) Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x, y, z) = (\sqrt{x^2 + y^2} - 10)^2 + z^2$$

  (a) Find the critical points of f.
  (b) Show that $\Sigma := f^{-1}(4)$ is a smooth 2-manifold.
  (c) Show that the intersection of the surface Σ with the 2-plane

$$\Pi := \{(x, y, z) \in \mathbb{R}^3 \mid z = 2\}$$

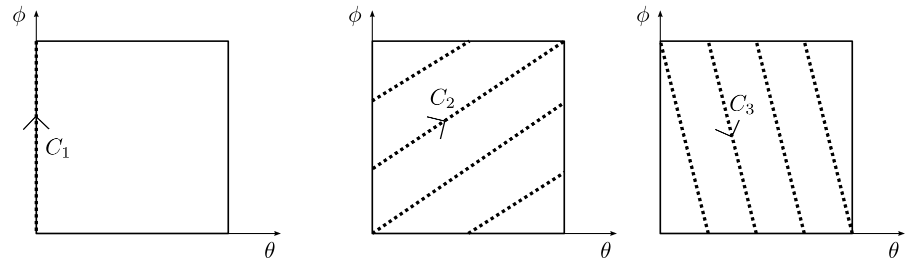
is *not* a transverse intersection. Is $S \cap \Pi$ a smooth manifold?

(5) Let $(\theta, \phi) \in S^1 \times S^1$ be two angular coordinates for the 2-torus $T^2 = S^1 \times S^1$, $\phi, \theta \in \mathbb{R}/(2\pi\mathbb{Z})$.

(a) Consider the following three closed curves in T^2:

$$C_1 := \{\theta = 0\}, \quad C_2 := \{2\theta = 3\phi\}, \quad C_3 := \{-4\theta = \phi\}$$

oriented as in this figure



We assume the orientation on the torus T^2 is the one induced on the quotient of the square from the standard orientation on $\mathbb{R}^2$ $(\partial_\theta, \partial_\phi)$. Find the three *oriented algebraic* intersection numbers $i(C_1, C_2)$, $i(C_1, C_3)$, and $i(C_2, C_3)$ between each pair of curves. Indicate whether each of the intersections is transverse.

(b) Compute the two integrals

$$\int_{C_3} d\theta, \quad \int_{C_3} d\phi.$$

(c) Show that the 1-forms $d\theta$ and $d\phi$ are not exact.

(6) Let A be an arbitrary symmetric 3×3 matrix. Consider the set

$$S = \left\{ (x, y, z) \mid \begin{pmatrix} x & y & z \end{pmatrix} A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 \right\}$$

Prove that S is either empty or it is a smooth 2-dimensional manifold.