# 185A Final Examination

**Thursday March 23, 2023**

Name______________________________ Student ID___________________ Signature_________________________

**Question 1:** Define the principal logarithm $\mathrm{Log}(z)$ (hint, start by writing $z$ in polar form). Explain how to use your definition to define the principal part of the power function $z^c$ where $c \in \mathbb{C}$. Use your definition to compute (the principal value of) $i^i$.

**Question 2:** *Draw* a regular pentagon in the complex plane with unit length sides and one side given by the line segment between $z = 0$ and $z = 1$.

*Factor* the polynomial

$$1 + z + z^2 + z^3 + z^4$$

over $\mathbb{C}$. Hint: use your pentagon for inspiration (you might want to compute the five complex numbers given by the differences of adjacent vertices).

*Extra space for question 2:*

**Question 3:** State the Cauchy–Riemann relations. Now consider the function

$$f(z) = x^3 + 3ix^2y - 3xy^2 + iy^3 \,.$$

Is $f(z)$ analytic?

*Extra space for question 3:*

**Question 4:** Let $1 < R \in \mathbb{R}$. Sketch the parameterized curves $C_1 = \{Re^{it} : t \in [0, \pi]\}$ and $C_2 = \{Rt : t \in [-1, 1]\}$.

Now consider the closed, simple, positively oriented contour $C = C_1 + C_2$. Use Cauchy's integral formula to compute the integral

$$\oint_C \frac{dz}{z^2 + 1}.$$

Hint: remember that polynomials can always be factored over $\mathbb{C}$.

Compute the limit

$$\lim_{R \to \infty} \frac{R}{R^2 - 1}.$$

Use the parameterization given to show

$$\left| \int_{C_1} \frac{dz}{z^2+1} \right| \leq \frac{\pi R}{R^2-1}.$$

Hint: you may need the reverse triangle inequality and may use any integral inequality proved in class.

Orchestrate your previous three calculations to compute the integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1}.$$

*Extra space for question 4:*

**Question 5:** *State* both Green's Theorem and the Cauchy–Goursat Theorem. Now assume that $C$ is a closed, simple, positively oriented contour. Does the Cauchy–Goursat theorem imply that the integral

$$A := \frac{1}{2i} \oint_C \bar{z} dz$$

is *zero*? If not, *why*? Now use Green's Theorem to *show* that $A$ equals the area inside the contour $C$.

*Extra space for question 5:*

**Question 6:** Suppose that the function $f(z)$ is analytic in a neighborhood of $z = 0$ and moreover

$$|f(z)| \leq |f(0)|$$

for all $z$ in that neighborhood. *Show* that $|f(z)|$ is a constant function in a neighborhood of $z = 0$. Hint: start by considering a closed, simple, positively oriented contour contained in a suitable neighborhood of the origin and then use Cauchy's integral formula.

*Extra space for question 6:*