MAT 21C SS-II
Final [100 points]
Time-2.5 hours

Instr: Neetal Neel

Choose any 5 out of 8 problems. Extra credit will be given for extra problems solved.

Each Problem is 20 points

- **It is a violation of the honour code to, in any way, assist another person or copy answers from any resources (physical (notes or books) or online). Please complete your own work and submit it on gradescope.**

  Any person found copying from any resources, if proved, will be granted a straight F.

  This course is under a strict no cheating policy.

- **No notes, books, calculators or online resources are allowed.**
- **Read directions to each problem carefully. Show full work to get full credit.In most**

cases, answer without a supporting work will receive little to no credit.

Exercise 1

1) [9]

Which of the sequences converge/diverge and why?

a)

$$\sqrt{n}\sin\left(\frac{1}{\sqrt{n}}\right)$$

b)

$$(3^n + 5^n)^{1/n}$$

c)

$$n\left(1 - \cos\left(\frac{1}{n}\right)\right)$$

2) [3+4+4=11]

Do the following series converge or diverge? State clearly the test you use. Also show why the series satisfies the assumptions(if any) the test requires.

a)

$$\sum_{n=1}^{\infty}\left(\cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)\right).$$

b)

$$\sum_{n=1}^{\infty}\frac{n\sqrt{n}}{n^2 - 1}$$

c)

$$\sum_{n=1}^{\infty} \frac{1.3.5.\ldots.(2n-1)}{2.4.6.\ldots.2n(3^n+1)}$$

Exercise 2

1) [8]

Find the interval of convergence and the radius of convergence for the series. Identify the values of x for which the series converges

a)conditionally

b) absolutely

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(4x-1)^n}{n^2}.$$

2) [6]

Let $f(x)$ be an even function, i.e

$$f(x)=f(-x);\ \text{for all } x.$$

Let

$$f(x)=\sum_{n=1}^{\infty}a_n x^n$$

is the Taylor series of x, and say it converges in the interval $(-R,R)$.

Prove that

$$a_{2n}=0 \text{ for } \quad ; n=0,1,2\ldots$$

3) [3]

Find the first 3 non-zero terms in the Maclaurin series for the following functions:

$$\sin x \tan^{-1}(x)$$

4) [3]

Do the following series converge or diverge? State clearly the test you use.

$$\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right).$$

Exercise 3

1) [6]

Use the Taylor series generated by $\ln x$ to find the limit:

$$\lim_{x\to 2} \frac{x^2-4}{\ln(x-1)}$$

2) [6]

If

$$\vec{a} = <1, 2, 3>; \vec{b} = <2, -3, 4>; \vec{c} = <3, 2, -2>$$

Let

$$\vec{d} = <1, 1, 1>$$

Write $\vec{d} = \vec{u_1} + \vec{u_2} + \vec{u_3}$ such that

$\vec{u_1}$ parallel to $\vec{a}$

$\vec{u_2}$ parallel to $\vec{b}$

$\vec{u_3}$ parallel to $\vec{c}$.

Find $\vec{u_1}, \vec{u_2}, \vec{u_3}$.

3) [6]

Express a vector in the form

$$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}.$$

Then express as a product of it's length and direction:

$$\overline{P_1P_2}: \qquad P_1(2,-3,-4); P_2(3,-2,6)$$

4) [2]

Write inequalities to describe the set:

The upper hemisphere of the sphere of radius 2 with centre at $(1,-1,2)$.

The North Pole is:

$(1,-1,4)$

and

the South pole:

$(1,-1,0)$

Exercise 4

1) [9]

MCQ (Might have multiple options correct) (+3 each) (Will receive full points only if all the correct options are chosen)

1. Which of the following are always true?
   a) Two orthogonal non-zero vectors have their dot product equal to 0.
   b)
   $$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{w} + \vec{u} \cdot \vec{v}.$$
   c)
   $$\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{w}) \cdot (\vec{u} \times \vec{v}).$$
   d) Two non-zero orthogonal vectors have the property that the magnitude of the cross product is equal to the product of the magnitudes of the individual vectors.
2. Which of the following are always true?
   a)
   $$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$$
   b)
   $$\vec{u} \times \vec{v} = 0$$
   implies $\vec{u}$ and $\vec{v}$ are non-zero parallel vectors.

c)

$$\text{proj}_{\vec{u}}(\vec{v}) \times \vec{v} = 0$$

d)

$$(\vec{u} \times \vec{v}) \cdot (\vec{u} + \vec{v}) = 0.$$

3. Consider the set

$$R = \{(x, y)|x^2 + y^2 > 4\}$$

a) R is closed and unbounded.

b) R is open, and does not contain $(1, \sqrt{3})$ in the interior

c) R is open, unbounded

d) R is unbounded and contains $(1, \sqrt{3})$ in the interior

2) [2]

Find the angle between the planes:

$$x + y + z = 5; \qquad 3x + y = 3$$

3) [2]

Find the angle between the lines:

$$L_1 : \quad x = 2 + 3t; y = 3 - t; z = \pi + t;$$

$$L_2 : \quad x = \pi + 2t; y = \ln 3 + t; z = 3 - 5t.$$

4) [4]

Find the area of the triangle:

$$(1, 0, -1), (1, 7, 2); (0, 3, 2)$$

5) [3]

How can we say that two planes

$$A_1x + B_1y + C_1z = D_1; \qquad A_2x + B_2y + C_2z = D_2$$

are parallel? Perpendicular? Give reasons for your answer

Exercise 5

1) [4]

Let $f(x, y)$ be a thrice differentiable function and let's say

$$H(a,b) = \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b). \end{bmatrix}$$

What happens if for a point (a, b)

$$det(H(ab)) > 0 \text{ and } f_{xx} = 0?$$

Justify your answer.

2) [8]

Find parametric equations of the following lines:

a) The line through $(2, 4, 5)$ perpendicular to the plane:

$$3x - 2y = 10.$$

b) The line through $(1, 2, 3)$ perpendicular to the vectors:

$$2\vec{i} + 3\vec{j} + 2\vec{k}; \qquad \vec{i} - 3\vec{j} + 3\vec{k}.$$

c) The line through $(1, -4, 0)$ and $(0, 0, 2)$

d) The line through the origin parallel to

$$< \pi, -e, 2.5 > .$$

3) [8]

MCQ (+2 each)

Only one option is correct.

1. Identify

$$R = \{(x, y, z) | x^2 + y^2 = 0\}$$

a) Infinite Cylinder
b) Circle
c) Line
d) Two points.

2. Identify

$$R = \{(x, y) | x^2 + y^2 \leq 0\}$$

a) Closed Disk
b) Open Disk
c) A Plane
d) Interior of an ellipse

3. Identify

$$R = \{(x, y, z) | 3x + 5y \geq 0; z = 2\}$$

a) One side of a plane
b) A line

c) A Plane

d) A line and a point.

4. Identify

$$R = \{(x, y)|y^2 - 5x + 6 = 0\}$$

a) Parabola

b) Hyperbola

c) A Plane

d) Ellipse

Exercise 6

1) [12]

Find the velocity and the position vectors as a function of time:

a)

$$\frac{d^2r}{dt^2} = \sin t\vec{i} + \cos t\vec{j} + (\cos^2 t - \sin^2 t)\vec{k}.$$

$$\left.\frac{dr}{dt}\right|_{t=0} = \sqrt{3}\vec{i} + \vec{j} - \vec{k}$$

$$r(\vec{0}) = 0.$$

b)

$$\frac{d^2r}{dt^2} = (t^3 + 3t)\vec{i} + 6\vec{k}$$

$$\left.\frac{dr}{dt}\right|_{t=0} = 0.$$

$$r(\vec{0}) = \vec{i} + 2\vec{j}.$$

2) [8]

A baseball is hit from the ground at an angle of 30° up from the horizontal. When and how far of the ground distance will the ball cover before it got caught in the stands which is at a height of $10ft$ from the ground. The initial speed is $10m/sec$? [Assume $g = 10m/sec^2$].

Also calulate the velocity at the instance of the ball getting caught.

Exercise 7

1) [4]

Let

$$w = f(s^2 + t^2); f''(x) = e^{2x}; f'(0) = \frac{1}{2}.$$

Find

$$\frac{\partial w}{\partial t} \text{ and } \frac{\partial w}{\partial s}.$$

2) [6]

Find ∇f and the equation of the tangent line to the given curve for the following:

a)

$$x^2 - y^2 = 1; \qquad (\sqrt{2}, 1).$$

b)

$$x^2 - xy + y^2 + 2x + y = 7; \qquad (-1, 2).$$

3) [6]

Find the local maxima, local minima and saddle points, of the following function in R:

$$z = f(x, y) = x^4 + y^4 + 4xy$$

Express your answer in triples (x_0, y_0, z_0).

4) [4]

Find the points on the sphere

$$x^2 + y^2 + z^2 = 25$$

where

$$f(x, y, z) = x + 2y + 3z$$

has its maximum and minimum values.

Exercise 8

1) [8]

Find all the first and second order partial derivatives of

$$z(x, y)$$

where x, y are independent variables at the point $(1, 1, 1)$:

$$xyz + z^3x - 2yz = 0.$$

2) [12]

Find the linearization of the function and an upper bound for the error. Please show all the work, along with full justifications

$$f(x, y, z) = \sqrt{2} \cos 2x \sin(2y + z);\ \text{at}\ P_0(0, 0, \pi/4)$$

$$R : |x| \leq 0.01; |y| \leq 0.01; \||z - \pi/4| \leq 0.01.$$

CheatSheet

- $$\sin 30 = \frac{1}{2}$$

- $$\cos 30 = \frac{\sqrt{3}}{2}$$

- $$\sin x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}; \qquad |x|\infty.$$

- $$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}; \qquad |x| \leq 1$$

- $$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}; \qquad -1 < x \leq 1.$$