MAT 280—Winter 2024
Moduli Spaces: Let's Feel, Touch, and Build Them!

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Aim of the course. In this course, we will study a few beautiful, deep, and elegant formulas associated with **moduli spaces**. Our key question is: "What is a moduli space, and how is it different from a usual space?"

In the first part, we study $\overline{\mathcal{M}}_{g,n}$, the moduli stack of stable algebraic curves of genus g with n smooth marked points. Instead of giving its full definition, our aim is to guide the participants to acquire a **working knowledge** of these spaces. This is similar to playing with Lego pieces, as appropriately phrased by Grothendieck: *dessins d'enfants*. It allows us to build moduli spaces through a combinatorial construction.

With this working knowledge at hand, in the second part, we study the moduli space of locally constant sheaves on any smooth curve, and the *Hitchin Systems*. Here, we extend our working knowledge to the concepts of $\mathcal{D}$-modules, sheaves and cohomologies.

At certain point, our working knowledge will lead us to the key property that every *moduli space* should possess: the existence of the *universal family*. When we formulate moduli spaces with this property as a *functor*, it becomes clear that manifolds and varieties cannot acquire such a property for most moduli problems. This is how *stacks* come in to our scope.

Constructions and formulas. We will approach the beautiful space $\overline{\mathcal{M}}_{g,n}$ by concretely constructing its open dense subspace $\mathcal{M}_{g,n}$ consisting of smooth curves. From our construction, i.e., putting dessins d'enfants together, we will see how the formula

$$\chi(\mathcal{M}_{g,n}) = (-1)^{(n-1)}\frac{(2g-3+n)!}{(2g-2)!}\cdot\zeta(1-2g), \quad 2g-2+n>0$$

holds. Here, $\chi(X)$ denotes the Euler characteristic of a topological space X, and $\zeta(s)$ is the Riemann zeta function. We will then calculate the generating function of intersection numbers of tautological cotangent classes of $H(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ from simple graph enumerations. I will explain how it allows us to understand the *Witten-Kontsevich formula* and the *Mirzakhani volume formula*.

A simple corollary to our construction is a characterization for a smooth compact Riemann surface to have a biholomorphic model as an algebraic curve *defined over* $\overline{\mathbb{Q}}$. This is essentially a topological characterization, unbelievably simple and elegant.

Another consideration along the same line of thoughts allows us to construct the moduli space of locally constant sheaves on an arbitrary smooth curve.

Categories, sheaves, cohomologies, etc. will be introduced as needs arise.

Prerequisite. Familiarity with undergraduate level experience in courses on Algebra, Lie Groups, Combinatorics/Discrete Mathematics, Real and Complex Analysis, Ordinary or Partial Differential Equations, and Number Theory are desirable. Knowledge on the basic materials covered in MAT 239 will be assumed.