MAT 22B, Final

Instructions:

- Unless otherwise stated, **justify all of your answers**. Partial credit will be given to partially correct answers. Answers with no justifications might not receive credit at all.
- Write all answers clearly in the provided space. Use reverse side of pages if needed.
- A scientific calculator is not allowed. It is not needed, you do not need to evaluate expressions of the form log(15).
- Textbooks, notes, and your own scratch paper are not allowed.

Name: ______________________________ **Id**: ______________________________

Name:

Problem 1. Solve an initial value problem

$$y' = ty^2, \quad y(0) = 1.$$

Name: ____

Problem 2. Let $y(t)$ be the solution of the initial value problem

$$y' = yt + 1, \quad y(0) = 1.$$

Compute $y(0.3)$ using the Euler method with a step $h = 0.1$.

Name: ____

Problem 3: Solve the initial value problem

$$y'' - 3y' - 4y = 0, \quad y(0) = 1,\, y'(0) = 0.$$

Name:

Problem 4: Researchers study the response of an oscillator to an applied force. They include friction in their modeling and model the displacement, $y(t)$, of the oscillator from it's equilibrium position by the equation

$$y'' + 2y' + 2y = \cos(\omega t),$$

where the parameter ω represents the frequency of the applied force. They use a mathematical computer program to solve the equation for three different values of ω, they use the same initial values in all three cases. They get solutions:

(i)

$$y(t) = -\frac{3}{5}e^{-t}\sin(t) + \frac{2}{5}\sin(t) - \frac{1}{5}e^{-t}\cos(t) + \frac{1}{5}\cos(t)$$

(ii)

$$y(t) = -\frac{9}{130}e^{-t}\sin(t) + \frac{4}{130}\sin(4t) + \frac{7}{130}e^{-t}\cos(t) - \frac{7}{130}\cos(4t)$$

(iii)

$$y(t) = -\frac{11}{85}e^{-t}\sin(t) + \frac{6}{85}\sin(3t) + \frac{7}{85}e^{-t}\cos(t) - \frac{7}{85}\cos(3t)$$

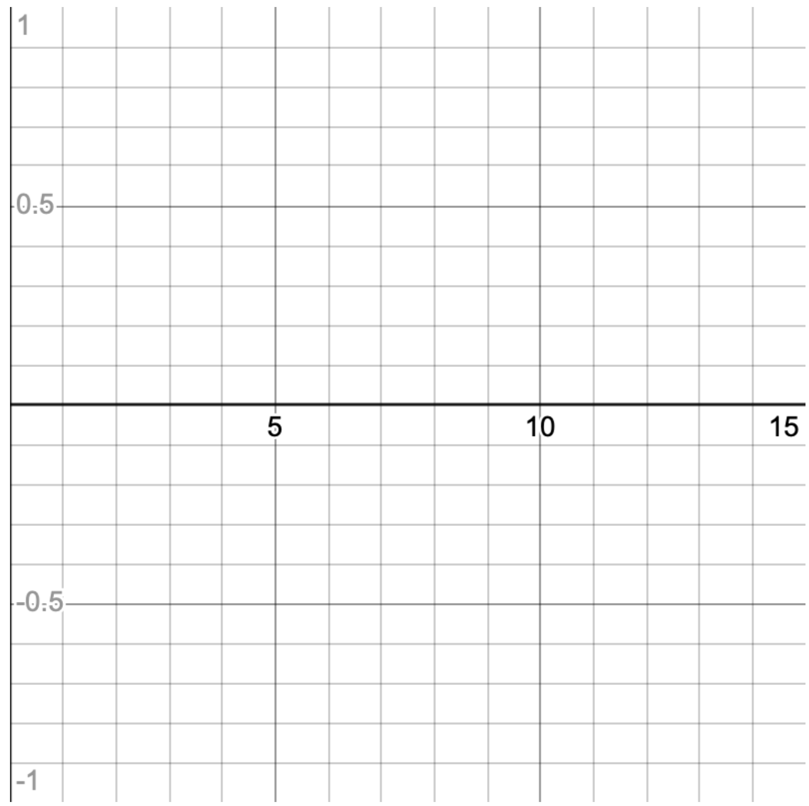
(a) Find a particular solution of the equation for $\omega = 1$.

(b) Determine the initial values that the researchers used.

Name:

(c) For each solution (i), (ii), (iii), determine the frequency ω of the applied force.

(d) Sketch the graph of the solution (i).



(e) Explain why, according to this model, the eventual (after a long time) frequency of oscillations of the oscillator is equal to the frequency of the applied force ω.

Name:

Problem 5: Solve the initial value problem

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Name: ____

Problem 6: Consider the system of differential equations

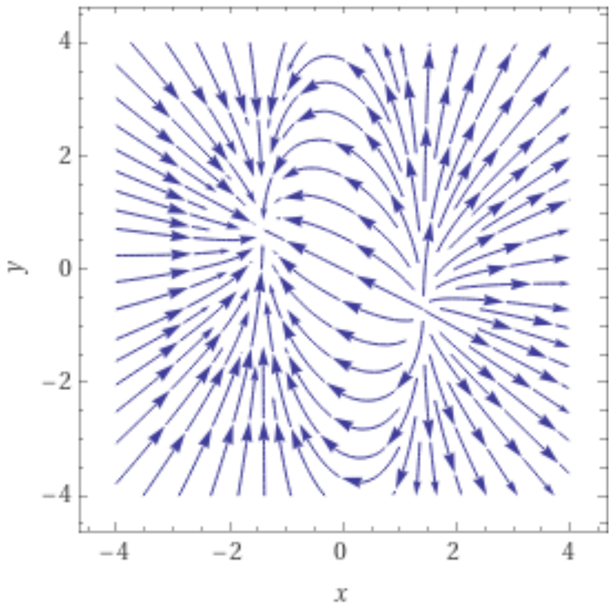
$$
\begin{aligned}
x_1' &= x_1 x_2 - 2 \\
x_2' &= x_1 - x_2 + 1
\end{aligned}
$$

(a) Find all equilibrium solutions for the system

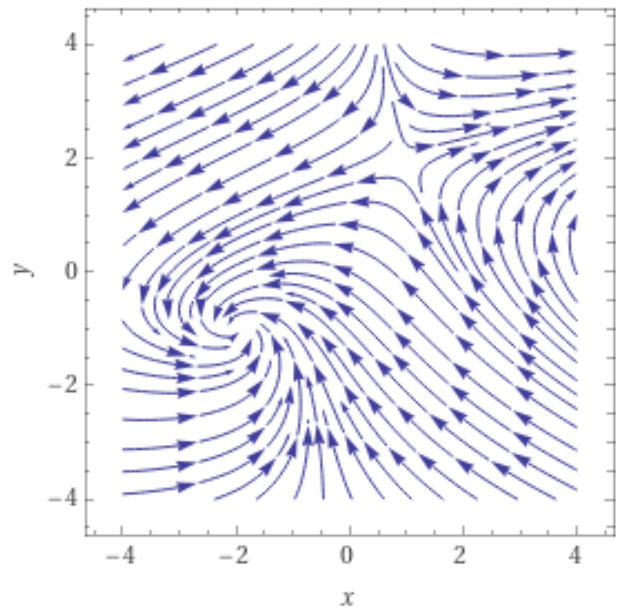
(b) Find the corresponding linearized system at each equilibrium point that you found in (a)

Name:

(c) Which of the plots below best represent the phase space plot of the system? (the x-axis in the plots is x_1, the y-axis in the plots is x_2)



1



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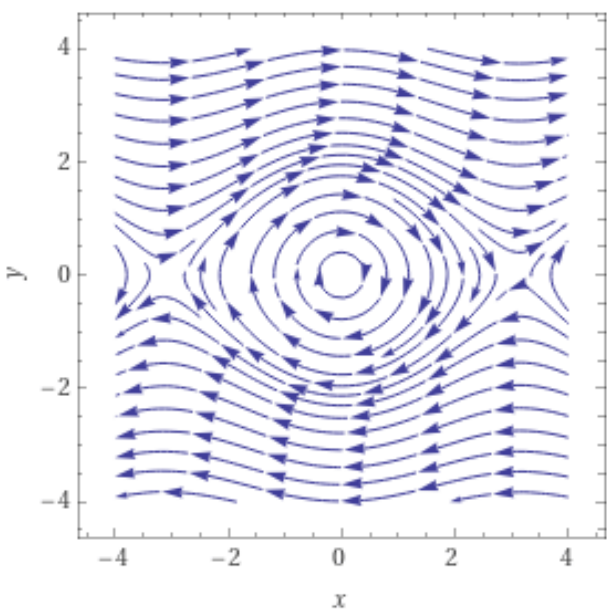

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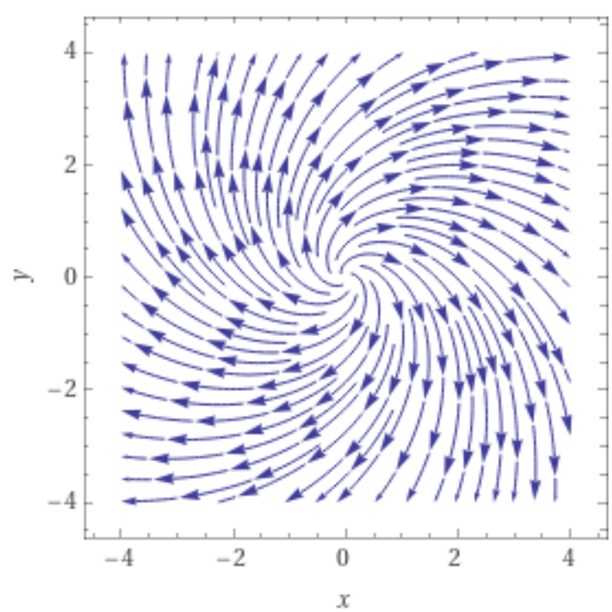


4

(d) Plot 1 above shows two equilibrium points, one stable and one unstable. Mark the stable with "S" and the unstable with "U". (You do not need to justify your answer).

Name:

Problem 7. Consider the system of equations

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

(a) Find the eigenvalues and eigenvectors of the matrix.

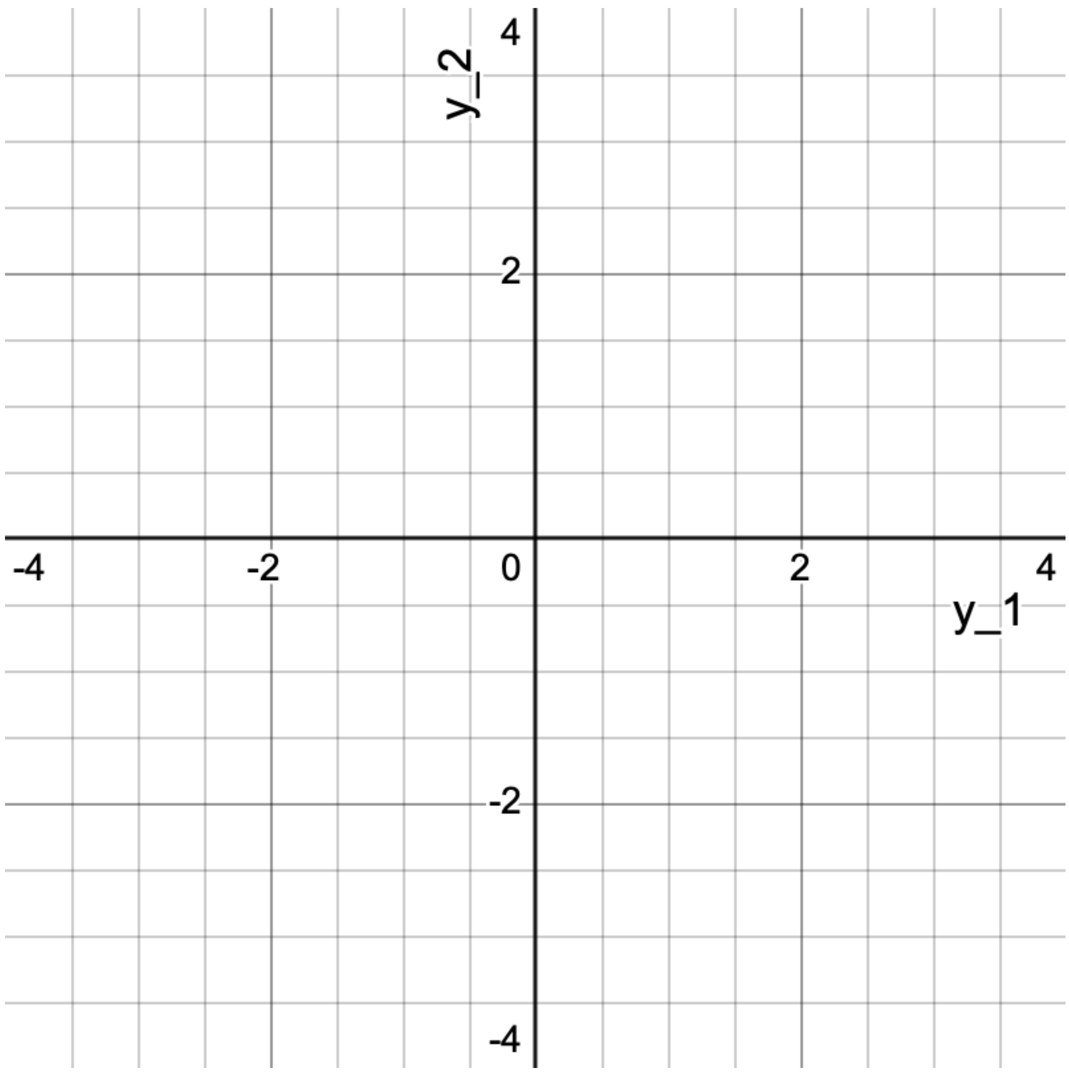
(b) Determine what kind of equilibrium point this is (for example: stable node, unstable node, stable spiral point, central point, etc).

Name:

(c) Sketch the phase space plot of the system. The plot should at minimum show

  (i) Direction arrows at points $(1, 2)$, $(2, 2)$, and $(-2, 2)$,

  (ii) Solutions passing through points $(1, 1), (-1, 3)$, and $(1, 2)$.



Name:

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