#1. (10 pts) Consider the matrix ODE
\[ M \frac{d^2u}{dt^2} + Au = 0 \]  
where \( u \in \mathbb{R}^N \), \( M \) is a \( N \times N \) diagonal matrix with positive entries, and \( A \) is a \( N \times N \) real symmetric matrix with positive eigenvalues. Both \( M \) and \( A \) are independent of \( t \). If we assume a solution of the form
\[ u(t) = e^{i\omega t}f \]
where \( f \in \mathbb{R}^N, f \neq 0 \), is independent of \( t \), show that \( \omega^2 \) must satisfy the equation
\[ \det (A - \omega^2 M) = 0. \]

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![Amplitude of Oscillation](image.png)

**Figure 1**: Figure for question #2. Amplitude as a function of driving frequency \( \omega \).

#2. (10 pts) A simple (classical) harmonic oscillator (i.e. a mass-spring system), initially at rest, is subjected to an external driving force \( F(t) = \cos \omega t \). The resulting amplitude of oscillation of the oscillator, as a function of the driving frequency \( \omega \), is displayed in Figure 1. Assuming that the frictional forces are very small; and hence negligible, on the basis of this graph estimate the period of the oscillator. Give a reason for your answer.
#3. (20 pts) The two-dimensional Laplace equation in polar coordinates \((x = r \cos \theta \text{ and } y = r \sin \theta)\) is

\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0. \tag{2}
\]

Assume a solution of the form (separation of variables)

\[u(r, \theta) = R(r)\Theta(\theta).\]

1. Find ODEs for \(R(r)\) and \(\Theta(\theta)\).
2. Solve the \(\Theta\) ODE. (You are not asked to solve the \(R\) ODE.)
3. What can you say about the separation constant, i.e. does it have to take on special values; and if so, why?

#4. (20 pts) Consider the nonlinear, coupled ODEs for the unknowns \(f_1(x), f_2(x)\) and \(f_3(x), x \in \mathbb{R}\):

\[
\begin{align*}
\frac{df_1}{dx} &= f_2(x) f_3(x), \\
\frac{df_2}{dx} &= -f_1(x) f_3(x), \\
\frac{df_3}{dx} &= -k^2 f_1(x) f_2(x),
\end{align*}
\]

with initial conditions \(f_1(0) = 0, f_2(0) = 1, f_3(0) = 1\) and \(k\) is a constant, \(0 \leq k \leq 1\). Show that

\[f_1(x)^2 + f_2(x)^2 = 1 \text{ and } k^2 f_1(x)^2 + f_3(x)^2 = 1\]

for all \(x\).

#5. (20 pts) Consider the 2nd order (scalar) ODE

\[
\frac{d^2 y}{dx^2} - x y = 0, \quad -\infty < x < \infty. \tag{3}
\]

Note that this is not a constant coefficient ODE.

1. We assume a power series solution of (3) of the form

\[y(x) = y_0 + y_1 x + y_2 x^2 + y_3 x^3 + \cdots = \sum_{n=0}^{\infty} y_n x^n \tag{4}\]

where the coefficients \(y_n\) are to be determined. Substitute (4) into (3) and show that

\[y_2 = 0 \tag{5}\]

and

\[(n + 2)(n + 1)y_{n+2} - y_{n-1} = 0 \quad \text{for } n = 1, 2, 3, \ldots. \tag{6}\]

Show that it follows from (5) and (6) that
1. \[ y_2 = y_5 = y_8 = y_{11} = \cdots = y_{3n+2} = \cdots \]

2. \[ y_{3n} = y_0 \prod_{j=0}^{n-1} \frac{1}{(3j+3)(3j+2)}, \quad n = 1, 2, 3, \ldots \]

3. \[ y_{3n+1} = y_1 \prod_{j=0}^{n-1} \frac{1}{(3j+4)(3j+3)}, \quad n = 1, 2, 3, \ldots \]

#6. (20 pts) In this problem you may assume as given

1. \[ \int_{-\infty}^{\infty} e^{-ax^2+2bx} \, dx = \sqrt{\frac{\pi}{a}} e^{b^2/a}, \quad a > 0. \]

2. If \( f(x) \) is continuous and rapidly decreasing at infinity, the Fourier transform of \( f \) is

\[ \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \, dx \]

and the inverse Fourier transform is

\[ f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi. \]

Consider now the one-dimensional heat equation on the line \((-\infty < x < \infty)\)

\[ \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0 \]

with initial condition

\[ u(x, 0) = f(x), \quad -\infty < x < \infty. \]

Show that the solution \( u(x,t) \) is given by

\[ u(x,t) = \int_{-\infty}^{\infty} K(x, y; t) f(y) \, dy \]

where

\[ K(x, y; t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}}. \]

END OF EXAM