Homework 1

due April 11

Problem 1. Prove that the following are matrix Lie groups:

(1) All invertible diagonal matrices.
(2) All invertible upper-triangular matrices.
(3) All unit upper-triangular matrices.

Problem 2. Let G be a matrix Lie group and G_0 the connected component of identity in G.

(1) Prove that G_0 is a normal subgroup of G.
(2) Prove that x and y are in the same connected component of G if and only if $x^{-1}y \in G_0$.
(3) Prove that all connected components of G are homeomorphic to G_0.

Note: you can use without proof that x and y are in the same connected component of G if and only if they are connected by a path.

Problem 3. Let a be an irrational real number and let G be the following subgroup of $\mathsf{GL}(2;\mathbb{C})$:

$$G = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{ita} \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

Show that

$$\overline{G} = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{is} \end{pmatrix} \mid t, s \in \mathbb{R} \right\},$$

where $\overline{G}$ denotes the closure of the set G inside the space of 2×2 matrices. Assume the following result: The set of numbers of the form $e^{2\pi i n a}$, $n \in \mathbb{Z}$, is dense in S^1.

Problem 4. *The group* $\mathsf{SU}(2)$. Show that if α and β are arbitrary complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$, then the matrix

$$A = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}$$

is in $\mathsf{SU}(2)$. Show that every $A \in \mathsf{SU}(2)$ can be expressed in this form for a unique pair (α, β) satisfying $|\alpha|^2 + |\beta|^2 = 1$. (Thus, $\mathsf{SU}(2)$ can be thought of as the three-dimensional sphere S^3 sitting inside $\mathbb{C}^2 = \mathbb{R}^4$. In particular, this shows that $\mathsf{SU}(2)$ is simply connected).

Problem 5. *The Heisenberg group.* Determine the center $Z(H)$ of the Heisenberg group H. Show that the quotient group $H/Z(H)$ is abelian.

Problem 6. Prove that $\mathsf{GL}(n;\mathbb{C})$ is connected in the following way.

Let $A, B \in \mathsf{GL}(n, \mathbb{C})$. Show that there are only finitely many $\lambda \in \mathbb{C}$ for which $\det(\lambda A + (1 - \lambda)B) = 0$. Show that there is a continuous path $A(t)$ of the form $A(t) = \lambda(t)A + (1 - \lambda(t))B$ connecting A to B and such that $A(t)$ lies in $\mathsf{GL}(n;\mathbb{C})$. Here, $\lambda(t)$ is a continuous path in the plane with $\lambda(0) = 0$ and $\lambda(1) = 1$.