MAT 261 University of California Spring 2025

Homework 2

due April 25

Problem 1. Let

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be the Pauli matrices.

(1) Show that the Pauli matrices form a basis over the real numbers of the vector space of 2×2 complex matrices X which are self-adjoint $X = X^*$ and traceless $\mathsf{Tr}(X) = 0$.
(2) Show that the Pauli matrices form an orthonormal basis under the inner product $\langle X, Y \rangle = \frac{1}{2}\mathsf{Tr}(XY)$.

Problem 2. Find the exponential of the following matrices:

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}.$$

Problem 3. Prove that

$$\exp \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \tag{1}$$

Problem 4. Show that every 2×2 matrix X with trace zero satisfies

$$X^2 = -\det(X) I.$$

Show that

$$\exp X = \cos\left(\sqrt{\det X}\right) I + \frac{\sin \sqrt{\det X}}{\sqrt{\det X}} X.$$

Use this to give an alternative derivation of (1).

Problem 5. Suppose that α is an irrational real number.

(1) Prove that $\Gamma_\alpha := \{e^{2\pi i \alpha k} : k \in \mathbb{Z}\}$ is a subgroup of $U(1) = S^1$.
(2) Prove that Γ_α contains elements of the form $e^{2\pi i \phi}$ with $0 < \phi < \frac{1}{N}$ for all N.
Hint: divide S^1 into N equal arcs and prove that one of them contains at least two elements of Γ_α.
(3) Use part (b) to prove that Γ_α is dense in $U(1)$ (and hence is **not** a matrix Lie group).

Problem 6. Suppose that α is an irrational real number. Use Problem 5 to prove that the one-parameter subgroup

$$\left\{ \exp \begin{pmatrix} 2\pi i t & 0 \\ 0 & 2\pi i \alpha t \end{pmatrix} : t \in \mathbb{R} \right\}$$

is dense in $U(1) \times U(1)$ (and hence is **not** a matrix Lie group).