WHAT ARE THE $n!$ AND $(n+1)^{n-1}$ THEOREMS?

INFORMAL SEMINAR

1. One simple example.

Let's begin with a warm-up. Consider the ring

$$R = k[x, y, z]$$

And the ideal:

$$I = \langle x + y + z, xy + yz + xz, xyz\rangle$$

Then R/I is a **finite** dimensional vectors space over k. A possible basis is given by the monomials

$$\{1, x, y, x^2, xy, x^2y\}$$

so it has dimension 6. Additionally, S_3 acts by permuting the variables in R and this induces a linear action on R/I, so this is a S_3 representation. Even more, the action respects the grading so we can decompose it degree by degree

0: The basis is 1 so the action is trivial. This corresponds to the partition $(1, 1, 1)$.
1: As an example, if we send x to z in the quotient this sends x to $-x - y$. This is the irreducible corresponding to partition $(2, 1)$.
2: This can be computed to be also the same irreducible as the previous one.
3: As an example, exchange x and y to get $x^2y \mapsto y^2x$, however

$$x^2y + y^2x = xy(x + y + z) - (xyz) \in I$$

So in the quotient $y^2x = -x^2y$. This means that it is the sign representation corresponding to the partition (3).

This implies that the whole representation on R/I is the regular one! But we achieved more, since we decomposed each graded piece. To summarize this information we consider the Frobenius series, representing each irred with the corresponding schur function, in this case we get

$$s_{(1,1,1)} + s_{(2,1)}q + s_{(2,1)}q^2 + s_{(3)}q^3$$

Sometimes priority is given to the schur function, so we write the above as

$$s_{(1,1,1)} + s_{(2,1)}(q + q^2) + s_{(3)}q^3$$

Moral is

We will have a *graded* representation and we want to decompose each graded piece.

2. The $n!$ statement.

Similar as above but things get more complicated as we get bigradings. Let's go through an example:

Fix $n = 3$ and partition $\lambda = (2,1)$. Consider the ring $k[x_1, x_2, x_3, y_1, y_2, y_3]$. We will consider the following determinant

$$\Delta_\lambda = \det[x_i^{p_j} y_i^{q_j}]$$

where (p_j, q_j) are the coordinates of the ferrers diagram of λ in french notation. In this case we get

$$\Delta_\lambda = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

And our vector space A_λ is the subspace spanned by all partial derivatives. In this case it is dimension 6 with basis

$$\{1, x_1 - x_2, x_2 - x_3, y_1 - y_2, y_2 - y_3, \Delta_\lambda\}$$

As before, this is a representation of S_3 respecting each **bi**graded piece. We are interested on each piece and in this case the Frobenius series is

$$s_{(1,1,1)} + s_{(2,1)}q + s_{(2,1)}t + s_{(3)}qt$$

.

The statements are:

(1) The dimension of A_λ is $n!$.
(2) Furthermore, it affords the regular representation of S_n.
(3) Even better, we know the Frobenius series. It is given by $\tilde{H}(\lambda; q, t)$, the transformed Macdonald polynomials.

Remark 1. The last item is innocently proving that $\tilde{H}(\lambda; q, t)$ is schur positive.

3. The $(n+1)^{(n-1)}$ statement.

This is easier to explain. More in the line of the first page, consider the ring $R = k[x_1, \cdots, x_n, y_1, \cdots, y_n]$ with the diagonal action of S_n and consider the ideal of invariant polynomials (without constant terms, so that the ideal is not the whole thing) I. The quotient R_n is a finite dimensional, bigraded representation of S_n. As in the $n!$ theorem we know:

(1) The dimension of R_n is $(n+1)^{(n-1)}$.
(2) Furthermore, as a representation of S_n it is isomorphic to $\epsilon \otimes PF$, where ϵ is the signed and PF is the permutation representation on parking functions.
(3) Even better, we know the Frobenius series. It's minimalistic expression is ∇e_n.

Remark 2. Again, the last item is innocently proving that ∇e_n is schur positive.