Math 127C Practice Final Answers Spring 2024

1. (Derivatives)
   Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ with $f((0,0)) = 0$ and otherwise
   $$f((x,y)) = \frac{xy}{x^2+y^2}.$$
   This function is differentiable at every point except $(0,0)$ where it is not.
   Find the following or show they do not exist:

   (a) the two by one total derivative matix $(Df)(1,2)$,

   (b) $f'((0,0);(1,2))$.

   ANS: See midterm solutions.

2. Assume that A is a bounded subset of $\mathbf{R}^n$ and $f : A \to \mathbf{R}$ is bounded. Assume further that for every $\epsilon > 0$ there are two finite collection of rectangles which together cover A and so that if R is in the first set then $M_R f - m_R f < \epsilon$ and the sum of volumes of those in the second set is at most ϵ.
   Show that $\int_A f$ exists.

   ANS: Directly: Fix a rectangle Q containing the (bounded) domain A of f and an interval $[-M, M]$ containing the (bounded) range of f. For each $\epsilon > 0$ choose two collections of rectangles as per the hypotheses along with a partition P of Q for which every subrectangle R associated to P is either disjoint from A or else contained in a rectangle from one (or both) of the collections. This can be done by taking the walls of the partition to be given by all the facets of rectangles in the finite collections which cut Q. Now compute $U_P f_A - L_P f_A = \sum_R (M_R f_A - m_R f_A) vR + \sum_S (M_S f_A - m_S f_A) vS + \sum_T (M_T f_A - m_T f_A) vT \leq \epsilon vQ + 2M\epsilon + 0$

where the three sums are over the rectangles associated to P with R contained in a rectangle from the first collectio,n, S contained in one from the second collection but not the first and T contained in neither (and hence not intersecting A). Thus $\overline{\int_A} f - \underline{\int_A} f < (vQ + 2M)\epsilon$ and the integral exists.

Rectifiability: It suffices to show that $\mu\mathrm{Disc}(f_A) = 0$....

3. Consider the region $S = \{(x,y)|0 < y < e^{\frac{-1}{x}}, x > 0\} \subseteq \mathbf{R}^2$.

   (a) Find a sequence of compact sets exhausting S. (That is $\{D_i\}$ with $D_i \subseteq \mathrm{Int}D_{i+1}$, each D_i compact and $\cup D_i = S$.)

   ANS: $D_n = \{(x,y)|\frac{1}{n} \leq y \leq e^{-x^{-1}} - \frac{1}{n}, \frac{1}{n} \leq x \leq n\}$.

   (b) Determine whether the extended (improper) integral $\int_S f$ exists if $f(x,y) = \frac{y}{x^2}$.

   ANS: The limt of the proper integrals over the regions D_n would work but since $f = \frac{y}{x^2}$ is bounded in the bounded open sets $U_n = \{(x,y)|0 < y < e^{-x^{-1}}, \frac{1}{n} < x < n\}$ the limit of the proper integrals over these regions also works. Computing $\int_{U_n} f = \frac{1}{4}[e^{\frac{-2}{n}} - e^{-n}]$ so $\lim_{n\to\infty} \int_{U_n} f = \frac{1}{4}$ exists.

4. If $f(x,y) = (x^2e^y, xy)$ and $g \circ f(x,y) = (x\cos(y), x\sin(y))$ find the matrix $[(Dg)(1,0)]$ and a neighborhood of this point on which g is locally invertible.

   ANS: First note that $f(1,0) = (1,0)$. Hence by the chain rule $[(D(g \circ f))(1,0)] = [(Dg)f(1,0)][(Df)(1,0)] = [(Dg)(1,0)][(Df)(1,0)]$. Direct computation gives $[(D(g \circ f))(1,0)] = \begin{bmatrix} \cos(y) & -x\sin(y) \\ \sin(y) & x\cos(y) \end{bmatrix}|_{(1,0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $[(Df)(1,0)] = \begin{bmatrix} 2xe^y & x^2e^y \\ y & x \end{bmatrix}|_{(1,0)} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$. Hence

$$[(Dg)(1,0)] = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}.$$

A neighborhood on which g is locally invertible would be one contained in an image under f of a neighborhood in which the determinants of the above two derivatives are nonzero. These determinants are x and $(2-y)x^2e^y$ which are nonzero in any box about $(1,0)$ of radius less than one such as $\frac{1}{2}$ and the image of this set under f is a sector of an anulus which contains (for instance) a box of radius $\frac{1}{10}$.

5. Consider a linear map $T_M : \mathbf{R}^n \to \mathbf{R}^n$ given by $T_M\overrightarrow{x} = [M]\overrightarrow{x}$. Show that if $A \subseteq \mathbf{R}^n$ and $v(A)$ exists then vT_MA also exists and $vT_MA = |\det[M]|vA$. Find an example in which $v(A \cup T_MA)$ exists even though vA does not.

   ANS: Write $B = T_MA$. If M is nonsingular then T_M is a diffeomorphism with (global) inverse $T_{M^{-1}}$. If vA exists then $vA = \int_A 1$ and A is rectifiable (bounded with measure zero boundary) and hence B is also rectifiable and by the change of variables formula $vB = \int_B 1 = \int_A (T_M \circ 1)|\det[DT_M] = \int_A |\det M|$.

   If M is not invertible then $\det M = 0$ and B is still rectifiable and is contained in the row space of M which has dimension less than n and hence after a linear diffeomorphism is a subset of the measure zero subset of $\mathbf{R}^n$ with $x_1 = 0$. Thus both numbers are zero.

   Finally an example set might be A the union of the rational points in $[-1,0]\times[-1,1]$ and the points with some irrational coordinate in $[0,1]\times[-1,1]$. The matrix M could be the rotation $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

6. (Partition of Unity)

(a) Show that if f and g are both differentiable at x then so is fg and $D(fg)(x) = f(Dg)(x) + g(Df)(x)$.

ANS: Direct computation: $\lim_{h\to 0} \frac{|f(h+x)g(h+x)-f(x)g(x)-f(x)[(Dg)(x)]h-[(Df)(x)]hg}{|h|}$
$= \lim_{h\to 0} \frac{|f(h+x)g(h+x)-f(h+x)g(x)+f(h+x)g(x)-f(x)g(x)-(f(h+x)-f(h+x)+f(x))[(Dg)(x)]h-}{|h|}$
$= \lim_{h\to 0} \frac{|f(h+x)(g(h+x)-g(x)-[(Dg)(x)]h)+(f(h+x)-f(x)-[(Df)(x)]h)+f(x))[(Dg)(x)]h)g(x)-(}{|h|}$
$\leq |f(x)| \lim_{h\to 0} \frac{|g(h+x)-g(x)-[(Dg)(x)]h|}{|h|}$
$+ \lim_{h\to 0} \frac{|f(h+x)-f(x)-[(Df)(x)]h)+f(x))[(Dg)(x)]h|}{|h|} g(x)$
$+ \lim_{h\to 0}(|f(x) - f(h+x)|)[(Dg)(x)]\frac{h}{|h|} = 0.$

Alternatively: Use the chain rule with $h : \mathbf{R}^n \to \mathbf{R}^2$ defined by $h(x) = (f(x), g(x))$ and multiplication $m : \mathbf{R}^2 \to \mathbf{R}$ defined by $m(u, v) = uv$. Thus $fg = m \circ h$ and by the chain rule $[(D(fg))(x)] = [(D(m\circ h))(x)] = [(Dm)(h(x))][(Dh)(x)] = [g(x), f(x)] \begin{bmatrix} (Df)(x) \\ (Dg)(x) \end{bmatrix}$ as desired since $[(Dm)(u, v)] = [v, u]$.

(b) Show that if $\{\phi_i\}$ is a class C^∞ partition of unity on A and $f : A \to \mathbf{R}$ is continuously differentiable on A (so $f = \sum_i \phi_i f$) then $Df = \sum_i \phi_i Df$.

ANS: Since $\sum \phi_i f(x) = f(x)$ and for each x only finitely many terms are nonzero $Df = D\sum \phi_i f = \sum D(\phi_i f) = (\sum D\phi_i)f + \sum \phi_i Df = (D(\sum \phi_i))f + \sum \phi_i Df = (D1)f + \sum \phi_i Df = \sum \phi_i Df$.

7. (a) Find a function $g : \mathbf{R}^2 \to \mathbf{R}^2$ which is an area preserving (that is if $A \subseteq \mathbf{R}^2$ is rectifiable then $vA = vgA$) class C^1 diffeomorphism and so that for every number R there are points $x, y \in \mathbf{R}^2$ with $|x - y| < 1$ and $|gx - gy| > R$.
Note that g can even be chosen to be primitive.

ANS: Note that by the change of variables formula if $|\det[Dg]| =$

1 at every point then g is area preserving since in this case $vgA = \int_{gA} 1 = \int_A g \circ 1|\det[Dg]| = \int_A 1$.

Accordingly choose $g(x, y) = (x, y + e^x)$ which is primitive (in the first variable) and has $[Dg] = \begin{bmatrix} 1 & 0 \\ e^x & 1 \end{bmatrix}$ which has determinant one and $|g(R, 0) - g(R+1, 0)| \geq e^{R+1} - e^R > e^R$.

(b) Show that for any such function $|Dg|$ must be unbounded.

ANS: By the mean value theorem $g(x) - g(y) = [(Dg)c](x - y)$ for some c between x and y. Thus if $|x - y| = 1$ and $|gx - gy| + R$ then $|[Dg](c)| \geq \frac{R}{2}$.