Math 127C Practice Midterm II Spring 2024

1. (Torsion) Consider the function

$$f((x,y,z)) = \begin{bmatrix} y\cos(x) \\ z\cos(y) \\ 2x\cos(z) \end{bmatrix}.$$

Find

$$(f\circ f\circ f)'\left((0,0,0); \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right).$$

ANS: Write $F = f\circ f\circ f$ and $\overrightarrow{0} = (0,0,0)$. Note that f and hence F are differentiable and hence $F'(\overrightarrow{0}; \overrightarrow{v}) = [(DF)\overrightarrow{0}][\overrightarrow{v}]$. Using the chain rule twice one gets

$$[(DF)\overrightarrow{0}] = [(Df)(f(f(\overrightarrow{0})))][(Df)(f(\overrightarrow{0}))][(Df)(\overrightarrow{0})].$$

Computing one gets that $f(\overrightarrow{0}) = \overrightarrow{0}$ and

$$[(Df)((x,y))] = \begin{bmatrix} -y\sin(x) & \cos(x) & 0 \\ 0 & -z\sin(y) & \cos(y) \\ 2\cos(z) & 0 & -2x\sin(z) \end{bmatrix}$$

so

$$[(Df)(\overrightarrow{0})]^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

and the answer is

$$\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

2. (IFT) Consider the function

$$f((x,y)) = \begin{bmatrix} x^2\sin(y) \\ x^2\cos(y) \end{bmatrix}.$$

Find a point in the plane which has a neighborhood on which f is invertible. Call this inverse function g. Choose a point (a,b) and find $(Dg)(a,b)$.

ANS: Compute

$$[(Df)((x,y))] = \begin{bmatrix} 2x\sin(y) & x^2\cos(y) \\ 2x\cos(y) & -x^2\sin(y) \end{bmatrix}$$

so $\det[(Df)((x,y))] = -2x^3$ is nonzero at $p = (1,0)$ (as well as many other points). Hence there is an open neighborhood $p \in U$ with $f(U) = V$ and an inverse function $g : V \to U$ to $f|_U : U \to V$ so $Id_U = g \circ f$ where $Id_U(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}$. Thus

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [(DId_U)(1,0)] = [(Dg \circ f|_U)(1,0)]$$

$$= [(Dg)(f(1,0))][(Df|_U)(1,0)] = [(Dg)(f(1,0))] \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}.$$

Finally taking $(a,b) = f(1,0) = (0,1)$ we get

$$[(Dg)(a,b)] = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{bmatrix}.$$

3. (Absolute) Show that if $f : Q \to \mathbb{R}$ is any bounded function on a rectangle then

$$\overline{\int_Q} f - \underline{\int_Q} f \geq \overline{\int_Q} |f| - \underline{\int_Q} |f|.$$

Use this to show that if f is integrable over Q then so is $|f|$.

ANS: Use the definitions to get the equalities:

$$\overline{\int_Q} f - \underline{\int_Q} f = \inf_P U_P f - \sup_{P'} L_{P'} f$$

$$= \inf_{P,P'} (U_P f - L_{P'} f) \geq \inf_{P,P'} (U_{P \cup P'} f - L_{P \cup P'} f)$$

$$= \inf_{P''} \left(\sum_R (M_R f - m_R f) \right) \geq \inf_{P''} \left(\sum_R (M_R |f| - m_R |f|) \right)$$

$$= \inf_{P''} (U_{P''} |f| - L_{P''} |f|) \geq \inf_{P'',P'''} (U_{P''} |f| - L_{P'''} |f|)$$

$$= \inf_{P''} (U_{P''} |f| - \sup_{P'''} L_{P'''} |f|) = \overline{\int_Q} |f| - \underline{\int_Q} |f|.$$

The first two inequalities hold because each term on the left is greater than or equal the corresponding term on the right. The last inequality holds because the left is an infimum over a smaller set.

4. (Ideal) Assume that $S \subseteq \mathbb{R}^n$ is bounded while $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ are both bounded and both continuous. Write $fg : S \to \mathbb{R}$ for the pointwise product function with $(fg)(x) = f(x)g(x)$. Show that if f is integrable over S then fg is also.

   ANS: By Theorem 13.5 it suffices to show that $E_{fg} = \{x_0 \in Bd(S) | \lim_{x \to x_0} f(x) \neq 0\}$ has measure zero. The set $Disc(f_S)$ has measure zero since f is integrable over S. Since f is continuous on S $Disc(f_S) = E_f$. Thus it suffices to show that $E_{fg} \subseteq E_f$ or equivalently (contrapositive) if $\lim_{x \to x_0} f(x) = 0$ then $\lim_{x \to x_0} fg(x) = 0$. Since g is bounded there is M with $-M \leq g(x) \leq M$ for every x and hence

$$-Mf(x) \leq g(x)f(x) \leq Mf(x)$$

   so if $\lim_{x \to x_0} f(x) = 0$ then $\lim_{x \to x_0} fg(x) = 0$ as required.