Math 17B Practice Midterm I
You may use one page of notes.
You may not use a calculator.
You may not use the textbook.
Please do not simplify answers.

1. (15 points) Find the volume of the solid obtained by rotating the region bounded by the curve \( y = 1 - x^2 \) and the \( x \)-axis about the \( x \)-axis.

**ANS:** For the bounds on \( x \) set the curves \( y = 1 - x^2 \) and \( y = 0 \) (the axis) equal: \( 1 - x^2 = 0 \) so \( 1 = x^2 \) so \( x = -1 \) and \( x = 1 \). Integrate:

\[
V = \pi \int_{x=-1}^{1} y^2 dx = \pi \int_{x=-1}^{1} (1-x^2)^2 dx = \pi \int_{x=-1}^{1} [1-2x^2+x^4] dx = \pi [x - \frac{2}{3}x^3 + \frac{1}{5}x^5]_{x=-1}^{1} = [1 - \frac{2}{3} + \frac{1}{5}] - [-1 + \frac{2}{3} - \frac{1}{5}].
\]

2. (20 points) In this problem \( F(t) \) is a function of the variable \( t \) and \( a \) is a parameter. Consider the differential equation

\[
\frac{dF}{dt} = a^2 t^2 F^2.
\]

Which of the following is a solution to the differential equation?

(a) i. \( F(t) = \frac{a^2}{t^3} \)

ii. \( F(t) = \frac{1}{2} e^{a^2 t^2} \)

iii. \( F(t) = \frac{a^2}{t^2} t^{-3} \)

**ANS:** (iii)

i. \( \frac{dF}{dt} = a^2 t^2 \neq a^2 t^2 F^2 \) so this is not a solution.

ii. \( \frac{dF}{dt} = a^2 t e^{a^2 t^2} \neq a^2 t^2 F^2 \) so this is not a solution.

iii. \( \frac{dF}{dt} = \frac{9}{a^2 t^2} \) and \( a^2 t^2 F^2 = a^2 t^2 \left( \frac{9}{a^2 t^2} \right) = \frac{9}{a^2 t^2} \) which are equal so this is a solution.

(b) Find a family of solutions to this differential equation. Your answer should involve both the parameter \( a \) and a constant of integration for the family.

**ANS:** Separate variables to get \( \frac{dF}{dt} = a^2 t^2 dt \) and integrate to get

\[
\int F^{-2} dF = a^2 \int t^2 dt \text{ and hence } -F^{-1} = a^2 \frac{t^3}{3} + C. \text{ Solve for } F(t) = \frac{1}{\frac{a^2 t^3}{3} + C}.
\]

3. (25 points) Consider the differential equation (the above one with \( a = 1 \)):

\[
\frac{dF}{dt} = t^2 F^2.
\]

(a) Find the solution to the equation if \( F(0) = 1 \).

**ANS:** Using the above solution with \( a = 1 \) gives \( 1 = F(0) = \frac{1}{\frac{1}{2} t^3 + C} = \frac{1}{C} \) so \( C = -1 \) and \( F(t) = \frac{1}{\frac{1}{2} t^3 - 1}. \)
(b) Use two steps of Euler’s method with step size $h = \frac{1}{2}$ to estimate $F(1)$ if (as in the previous part) $F(0) = 1$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$F$</th>
<th>$\frac{dF}{dt}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>$1 + \frac{1}{2}(0)$</td>
<td>$(\frac{1}{2})^2$</td>
</tr>
<tr>
<td>1</td>
<td>$1 + \frac{1}{2}(\frac{1}{2})^2$</td>
<td>$\ldots$</td>
</tr>
</tbody>
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Thus $F(1) \sim 1 + \frac{1}{8}$.

4. (15 points) If $\frac{dF}{dt} = -(F - 1)(F - 2)^2(F - 3)$

(a) Find the equilibrium values for $F$.

**ANS:** $F$ is at equilibrium if $\frac{dF}{dt} = 0$ which occurs if $F = 1$, $F = 2$ or $F = 3$.

(b) Determine which (if any) are stable.

**ANS:** If $F < 1$ then $\frac{dF}{dt} < 0$ so $F$ is decreasing.
If $F < 1$ then $\frac{dF}{dt} < 0$ so $F$ is decreasing.
If $1 < F < 2$ then $\frac{dF}{dt} > 0$ so $F$ is increasing.
If $2 < F < 3$ then $\frac{dF}{dt} < 0$ so $F$ is increasing.
If $3 < F$ then $\frac{dF}{dt} < 0$ so $F$ is decreasing.
Thus $F = 3$ is the only stable equilibrium.

Note that the function $g(x) = -(x - 1)(x - 2)^2(x - 3)$ is nonnegative if $x$ is between one and three.
5. (25 points) Consider the Lotka-Volterra type system of differential equations with \( F(t) \) the population of F as a function of time and \( G(t) \) the population of G as a function of time:

\[
\frac{dF}{dt} = F(1000 - G) \\
\frac{dG}{dt} = G(2000 - F)
\]

(a) Decide and explain which scenario this system represents:
   i. \( F \) and \( G \) cooperate.
   ii. \( F \) and \( G \) compete.
   iii. \( F \) eats \( G \).
   iv. \( G \) eats \( F \).

**ANS:** (ii): The \( FG \) term in \( \frac{dF}{dt} \) is \(-FG\) and the \( FG \) term in \( \frac{dG}{dt} \) is also \(-FG\) so for each of them interaction with the other causes a decrease and the relationship is competitive.

(b) Sketch and label the lines in the G-F plane along which the direction field for the associated nonautonomous differential equation

\[
\frac{dF}{dG} = \frac{F(1000 - G)}{G(2000 - F)}
\]

is vertical and the lines along which it is horizontal (the nullclines).

**ANS:** The direction field is horizontal along the lines \( F = 0 \) and \( G = 1000 \) (\( F \)-nullclines). It is vertical along the lines \( G = 0 \) and \( F = 2000 \) (\( G \)-nullclines).

(c) Using your graph from the previous part indicate with arrows whether \( F \) and \( G \) are increasing, decreasing or constant in each of the regions your lines created.

**ANS:** The four lines divide the plane into nine regions. In four of these \( G \) and \( F \) are positive and for these select a point in each: (500, 500), (3000, 500), (500, 3000) and (3000, 3000) and at these four points the pairs (sign of \( \frac{dG}{dt} \), sign of \( \frac{dF}{dt} \)) are (+,+), (+,-), (-,+), and (-,-) respectively.

(d) Find any stable equilibrium points of the system.

**ANS:** Thus two of the four arrows point away from the point where the lines cross with \( F = 2000 \) and \( G = 1000 \) so this is an unstable equilibrium. Similarly the arrow points away from the point with \( G = F = 0 \) which is the other unstable equilibrium.