Math 21C Winter 2018 Practice Final Answers, March 22.
To receive full credit you must show all of your work. Please do not use any cell phones, notes or books. A calculator is allowed. Do not simplify your solutions

1. Determine whether the following limit exists.

$$\lim_{n\to\infty} \frac{10^{6n}}{\sqrt{n^n}}$$

ANS: $\lim_{n\to\infty} \frac{10^{6n}}{\sqrt{n^n}} = \lim_{n\to\infty} (\frac{10^6}{\sqrt{n}})^n$ converges to 0.

2. How many terms of the following convergent series should be used to estimate its value with an error of at most 0.1?

$$\sum_{n=1}^{\infty} 2ne^{-n^2}$$

ANS: Using the integral test error estimate gives $R_n \leq \int_n^\infty 2xe^{-x^2}dx = e^{-n^2}$ so if $n = 2$ then $R_n \leq \frac{1}{e^4} < \frac{1}{10}$. Thus the first two terms suffice.

3. Use the alternating series error bound to estimate the error from using the first two nonzero terms in the Maclaurin series for e^{-x^2} to estimate the definite integral

$$\int_0^{\frac{1}{10}} e^{-x^2} dx.$$

ANS: The first three nonzero terms of the series are $e^{-x^2} = 1 - x^2 + \frac{1}{2}x^4$. The integrals of the terms of the series alternate in sign and decrease in absolute value to zero so the absolute value of the integral of the third term (which is $\int_0^{0.1} \frac{1}{2}x^4 dx = 10^{-6}$) is a bound on the error from using the first two.

4. Determine whether the following series converges.

$$\sum_{n=1}^{\infty} \frac{1}{\ln(n)\sqrt{n}}$$

ANS: $\lim_{n\to\infty} \frac{\ln(n)}{n^{\frac{1}{4}}} = \lim_{n\to\infty} \frac{4n^{-1}}{n^{\frac{-3}{4}}} = 0$ by l'Hospital's rule so $\frac{\ln(n)}{n^{\frac{1}{4}}}$ is eventually less than one and by the limit comparison test $\sum_{n=1}^{\infty} \frac{1}{\ln(n)\sqrt{n}} \geq \sum_{n=1}^{\infty} \frac{\ln(n)}{n^{\frac{1}{4}}} \frac{1}{\ln(n)\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{4}}}$ which diverges by the p-series test so the given series also diverges.

5. Determine the values of x for which the following series converges. Be sure to check the end points of the interval.

$$\sum_{n=0}^{\infty} \frac{(2-x)^n}{\sqrt{n}}$$

ANS: By the ratio test the series converges (in fact absolutely) if $|\lim(2-x)\sqrt{\frac{n}{n+1}}| = |2-x| < 1$ or $-1 < 2-x < 1$ or $3 > x > 1$ and diverges if $|\lim(2-x)\sqrt{\frac{n}{n+1}}| > 1$ or either $x > 3$ or $x < 1$. If $x = 1$ the series diverges by the p-series teest and if $x = 3$ it converges by the alternating series test.

6. Find the first three nonzero terms of the Taylor series about $x = 0$ for the following function.

$$\frac{1+x^2}{1-2x}$$

ANS: $(1+x^2)(1+2x+4x^2+\ldots) = 1+2x+5x^2+\ldots$.

7. Find an equation for the set of points in space which are the same distance from the origin as they are from the point $(1,1,1)$.

ANS: $\sqrt{x^2+y^2+z^2} = \sqrt{(x-1)^2+(y-1)^2+(z-1)^2}$ or $x+y+z = \frac{3}{2}$.

8. Find the cosine of any one of the angles in the triangle with corners at $(1,1,1)$, $(1,2,3)$ and $(5,4,3)$.

ANS: $\cos(\theta) = \frac{\overline{P_1P_0}\cdot\overline{P_1P_2}}{|\overline{P_1P_0}||\overline{P_1P_2}|} = \frac{\langle 0,-1,-2\rangle\cdot\langle 4,2,0\rangle}{\sqrt{5}\sqrt{20}} = -0.2$.

9. Find an equation for the plane containing the points $(1,1,1)$, $(1,2,3)$ and $(5,4,3)$.

ANS: $\overline{P_1P_0} \times \overline{P_1P_2} = \langle 4,-8,4\rangle$ is a normal vector to the plane so an equation is $4x-8y+4z = c$ and checking any of the three points gives $c = 0$.

10. If the distance between the points P and Q is 5 and the distance between the point P and the line $Q + t\mathbf{u}$ is 4 what is the distance between the point P and the plane through Q and perpendicular to $\mathbf{u}$?

ANS: The points P, Q and the closest point on the plane to P form a right triangle so the distance is $\sqrt{5^2-4^2} = 3$.

11. Show that the following limit does not exist.

$$\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2}$$

ANS: Along the path with $y = 0$ the limit s 0. Along the path with $y = x - 1$ the limit is $\frac{1}{2}$.

12. Find $\frac{dx}{ds}$ at $s = \frac{\pi}{2}$ if $w = x^2y^2$, $y(s) = \sin(s)$, $x(s)$ is unknown but $x(\frac{\pi}{2}) = 3$ and $\frac{dw}{ds} = 2$ at $s = \frac{\pi}{2}$. (There was a typo in the practice exam with 1 occuring instead of $\frac{\pi}{2}$ in two places).

    ANS: $\frac{dw}{ds} = \frac{\partial w}{\partial x}\frac{dx}{ds} + \frac{\partial w}{\partial y}\frac{dy}{ds}$ and $\frac{\partial w}{\partial x} = 2xy^2$, $\frac{\partial w}{\partial y} = 2x^2y$, $\frac{dy}{ds} = \cos(s)$. At the point where $s = \frac{\pi}{2}$ we have $\frac{dw}{ds} = 2$, $x = 3$, $y = \sin(\frac{\pi}{2}) = \frac{1}{\sqrt{2}}$ and $\frac{dy}{ds} = \cos(\frac{\pi}{2}) = \frac{1}{\sqrt{2}}$. Thus $2 = 3\frac{dx}{ds} + 9$ so $\frac{x}{s} = \frac{-7}{3}$.

13. Find the two directions in which the directional derivative of $f(x, y)$ at the point $(1, 1)$ is zero.

$$f(x, y) = x^2y - y^2x$$

    ANS: $\nabla f(1, 1) = \langle 2 - 1, 1 - 2\rangle$ and $D_{\mathbf{u}}f(1, 1) = \langle 1, -1\rangle \cdot \mathbf{u}$ which is 0 for $\mathbf{u} = \pm\frac{1}{\sqrt{2}}\langle 1, 1\rangle$.

14. Find the point on the elliptic paraboloid given by $z = 1 - x^2 - y^2$ which is closest to the point $(0, 0, 2)$.

    ANS: Use Lagrange multipliers with $f(x, y, z) = x^2 + y^2 + (z - 2)^2$ and $g(x, y, z) = x^2 + y^2 + z - 1$ so that $\nabla f = \langle 2x, 2y, 2z - 4\rangle = \lambda\nabla g = \lambda\langle 2x, 2y, 1\rangle$ so $2z - 4 = \lambda$ and $z = \frac{-3}{2}$ or $x = y = 0$. In the first case there are no solutions and in the second the only solution is $(0, 0, 1)$. Since there is a closest point and it is a solution to the Lagrange multiplier equations but there is only one solution it must be the closest point.

15. Find and classify the critical point(s) and find the maximum value of the function $y^2 - x^2$ on the circular disk given by $4(x - 1)^2 + 4y^2 \leq 9$.

    ANS: Write $f(x, y) = x^2 - y^2$. Since $\nabla(f) = \langle 2x, -2y\rangle$ is zero only at $(0, 0)$ this is the only critical point. At this point $f_{xx} = 1$, $f_{yy} = -1$ and $f_{xy} = f_{yx} = 0$ one has $f_{xx}f_{yy} - f_{xy}^2 < 0$ so $(0, 0)$ is a saddle point.

    The maximum is therefore on the boundary circle so use Lagrange multipliers to find it with $g(x, y) = 4(x - 1)^2 + 4y^2 - 9$ so $\nabla(f) = \langle 2x, -2y\rangle = \lambda\nabla g = \lambda\langle 8x-8, 8y\rangle$ and either $y = 0$ or $\lambda = -\frac{1}{4}$. If $y = 0$ then $(x-1)^2 = \frac{9}{4}$ so the points are $(\frac{5}{2}, 0)$ and $(\frac{-1}{2}, 0)$. If $\lambda = -\frac{1}{4}$ then $2x = 2 - 2x$ so $x = \frac{1}{2}$ and the points are $(\frac{1}{2}, \sqrt{2})$ and $(\frac{1}{2}, -\sqrt{2})$. Checking all four points 3he first two give negative values for f and the last two both give the maximum value of $\frac{7}{4}$.

16. Find the largest area of a rectangle with corners on the ellipse $\frac{x^2}{16} + y^2 = 1$ and edges parallel to the x and y axes.

    ANS: Take x and y positive and use Lagrange multipliers with $f(x, y) = 4xy$ and $g(x, y) = \frac{x^2}{16} + y^2 - 1$ so that $\nabla f = \langle 4y, 4x\rangle = \lambda\nabla g = \lambda\langle\frac{x}{8}, 2y\rangle$ so that $4y = \frac{\lambda}{8}x = \frac{\lambda^2}{16}y$ and $\lambda = 8$ or $y = 0$. If $y = 0$ the area is 0. If $\lambda = 8$ then $x = 4y$ and $y = \frac{1}{\sqrt{2}}$ and the maximum area is 8.

17. (Optional extra credit problem.) Find the value of a for which the following limit exists and find the limit.

$$\lim_{x \to 0} \frac{\sin(ax) - \sin(x) - x}{x^3}$$

ANS: Consider the Taylor's series $\sin(ax) - \sin(x) - x = (a-2)x - \frac{a^3-1}{6}x^3 + \ldots$. Thus $a = 2$ and the limit is $\frac{7}{6}$.