Math 21C Final Answers Spring 2024 You may use one page of notes but not a calculator or textbook.
Please do not simplify your answers.

1. (23 points) (Limits)
   Determine whether the following series converge.
   Specify the test(s) you use.

   (a) $\sum_{n=1}^{\infty} \frac{3^n+(-4)^n}{5^n}$
   ANS: Converges since it is a sum of two congergent geometric series. In fact $\sum_{n=1}^{\infty} \frac{3^n}{5^n} + \sum_{n=1}^{\infty} \frac{(-4)^n}{5^n} = \frac{3}{5}\frac{1}{1-\frac{3}{5}} + \frac{-4}{5}\frac{1}{1-\frac{-4}{5}}$.

   (b) $\sum_{n=1}^{\infty} \frac{n+\sqrt{n}}{n^2+n}$
   ANS: Diverges by limit comparison to the divergent p-series $\sum \frac{1}{n}$. Computing $\lim_{n\to\infty} \frac{n+\sqrt{n}}{n^2+n} n = \lim_{n\to\infty} \frac{1+n^{\frac{-1}{2}}}{1+n^{-1}} = 1$ so the given sum diverges if (and only if) the comparison sum $\sum_{n=1}^{\infty} \frac{1}{n}$ does.

2. (24 points) (Error Bounds)
   For each of the following find an upper bound for the error resulting from estimating the infinite sum using just the first five terms. Be sure to check that any functions that need to be decreasing in fact are decreasing.

   (a) Use an integral test error bound. $\sum_{n=1}^{\infty} ne^{\frac{-n^2}{2}}$
   ANS: Consider the function $f(x) = xe^{\frac{-x^2}{2}}$ so the terms of the series are $f(n)$ and compute $f'(x) = (1-x^2)e^{\frac{-x^2}{2}} \leq 0$ if $x \geq 1$ so the terms are positive and decreasing. Thus the required error is bounded by $|\int_5^{\infty} f(x)dx| = | - e^{\frac{-x^2}{2}}|_5^{\infty}| = e^{\frac{-25}{2}}$.

   (b) Use the alternating test error bound. $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{\sqrt{n^2+1}}$
   ANS: The terms of the series are alternating in sign and are $\sqrt{\frac{1}{n+n^{-1}}}$ which is a square root of one over an increasing function and hence decreasing and has limit zero. Thus the alternating series bound applies and the reeor is bounded by the sixth term: $|\sqrt{\frac{6}{36+1}}|$.

3. (17 points)(Interval of Convergence)
   Determine the values of x for which the following series converges.
   Be sure to check the end points of the interval. $\sum_{n=1}^{\infty} \frac{x^n}{3^n+4^n}$
   ANS: By the ratio test the series converges (absolutely) if $|\lim_{n\to\infty} \frac{(3^{n+1}+4^{n+1})x^n}{(3^n+4^n)x^{n+1}}| = |\lim_{n\to\infty} \frac{(3\frac{3}{4}^n+4)}{(\frac{3}{4}^n+1)x}| = |\frac{4}{x}| > 1$ which occurs if $-4 < x < 4$. By the same test it diverges if the same limit has absolute value less than one which occurs if $x < -4$ or $x > 4$.
   If $x = \pm 4$ the series divernes by the n-th term test since $\lim_{n\to\infty} \frac{4^n}{3^n+4^n} = 1$ and $\lim_{n\to\infty} \frac{(-4)^n}{3^n+4^n}$ does not exist (in particular neither is zero).

4. (17 points)(Taylor)
Find the first three nonzero terms of the Taylor series about $x = 0$ for the following function. $\frac{x}{1+x} + e^{x^2}$
ANS: Expanding gives $x(1-x+x^2-x^3+\ldots)+(1+(x^2)+\frac{1}{2}(x^2)^2+\ldots) = 1+x-x^2+x^2+x^3-x^4+\frac{1}{2}x^4\ldots$ so the first three nonzero terms add to $1+x+x^3$.

5. (17 points Vectors) Find the area of the triangle with corners at $P(1,2,3)$, $Q(1,2,4)$ and $R(2,3,4)$.
ANS: The area is half the length of the cross product of any two of $\overline{PQ}$, $\overline{QR}$, $\overline{RP}$ so use the first two and compute $\overline{PQ} \times \overline{QR} = \langle -1,1,0\rangle$ which has length $\sqrt{2}$ so the area is $\frac{\sqrt{2}}{2}$.

6. (17 points) (Line) Find parametric equations for the line through the point $P(1,2,3)$ and normal to the plane $z = 5x - 3y + 10$.
ANS: The plane equation can be written as $0 = 5x - 3y - z + 10$ which has $\langle 5,-3,-1\rangle$ as a normal vector so the parametric equations $x = 1+5t$, $y = 2-3t$ and $z = 3-t$ give the required line.

7. (17 points)(Surface) Find an equation for the plane tangent at the point $P(0,1,2)$ to the surface given by the equation $2e^x = y^2 z$
ANS: Write $0 = g(x,y,z) = 2e^x - y^2 z$ for the equation of the surface which has gradient $\vec{\nabla} g = \langle 2e^x, -2yz, -y^2\rangle$ which at the given point is $\langle 2,-4,-1\rangle$ so an equation for the tangent plane is $0 = 2x - 4(y-1) - (z-1)$.

8. (17 points)(Limits) Show that the following limit does not exist.
$\lim_{(x,y)\to(0,0)} \frac{(x+y)^4}{x^4+(xy)^2+y^4}$
ANS: Use the two line test. For one line take $y = 0$ and compute
$\lim_{(x,0)\to(0,0)} \frac{(x+0)^4}{x^4} = 1$. For the other take $y = x$ and compute
$\lim_{(x,x)\to(0,0)} \frac{(x+x)^4}{x^4+(xx)^2+x^4} = \frac{16}{3}$. These two limits along line differ so the limit does not exist.

9. (17 points) (Chain Rule)
Assume that $w = x^2 y$ is a function of x and y. Assume further that $y = t^2$ is a function of t and x is some unknown function of t.
You know that if $t = 2$ then $x = 1$ and $\frac{dw}{dt} = 20$.
Find $\frac{dx}{dt}$ if $t = 2$.
ANS: By the chain rule $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$. Computing directly $\frac{dy}{dt} = 2t$, $\frac{\partial w}{\partial x} = 2xy$, and $\frac{\partial w}{\partial y} = x^2$. At $t = 2$ there is $x = 1$, $y = 4$, $\frac{\partial w}{\partial x} = 8$, $\frac{\partial w}{\partial y} = 4$ and $\frac{dy}{dt} = 4$. Combining these at $t = 2$ gives $20 = 8\frac{dx}{dt} + (4)(4)$ so $\frac{dx}{dt} = \frac{4}{8}$.

10. (17 points)(Critical) Consider the function $f(x,y) = x^2 + xy^2 + y^3$.

    (a) Find all the critical points on the graph of the function.
    ANS: Critical points are those with $f_x = f_y = 0$. Compute $f_x = 2x + y^2$ and $f_y = 2xy + 3y^2$. Solve using the first to get $x = \frac{-1}{2}y^2$

and use it to eliminate x from the second to get $0 = 2\frac{-1}{2}y^2y + 3y^2 = y^2(3-y)$. This has solutions $y = 0$ and $y = 3$ and the formula for x gives two critical points at $(0,0)$ and $(\frac{-9}{2}, 3)$.

(b) Use the second derivative test to identify one saddle point.
ANS: The second derivative test identifies a critical point as a saddle point if $f_{xx}f_{yy} - f_{xy}^2 < 0$. Computing gives $f_{xx} = 2$, $f_{xy} = 2y$ and $f_{yy} = 2x + 6y$ and hence $f_{xx}f_{yy} - f_{xy}^2 = 4y - 4(x+3y)^2$. At the first critical point this value is zero and the test is inconclusive. At the second it is negative so $(\frac{-9}{2}, 3)$ is certified to ba a saddle point.

11. (17 points)(Constrained Optima) Find the Maximum value of $x^2 + y^2$ subject to the constraint that $x^2 - 2x + y^2 - 6y = 0$.
ANS: This is a constrained optimization problem for which a Lagrange multiplier is suitable with $f(x,y) = x^2 + y^2$ the function to optimize and $0 = g(x,y) = x^2 - 2x + y^2 - 6y$ the constraint. The method gives that every local constrained optimum satisfies first that the directions of $\overrightarrow{\nabla} f$ and $\overrightarrow{\nabla} g$ are the same (or equivalently $\lambda\overrightarrow{\nabla} f = \overrightarrow{\nabla} f$ for some λ) and the constraint $0 = g$ holds. Computing $\overrightarrow{\nabla} f = \langle 2x, 2y\rangle$ and $\overrightarrow{\nabla} g = \langle 2x-2, 2y-6\rangle$. Hence the three equations to solve for possible optima are $\lambda 2x = 2x - 2$, $\lambda 2y = 2y - 6$ and $0 = x^2 - 2x + y^2 - 6y$. Use up the first to solve for $\lambda = \frac{x-1}{x}$ (unless $x = 0$) and substitute this into the second to get $\frac{x-1}{x}2y = 2y - 6$ or equivalently $2y(x-1) = x(2y-6)$ or equivalently $2xy - 2y = 2xy - 6x$ or equivalently $y = 3x$. Finally use this result to substitute into the third equation and get $0 = x^2 - 2x + 9x^2 - 18x = 10x(x-2)$ which has solutions $x = 0$ and $x = 2$ and hence $y = 0$ or $y = 6$. Among these the function (x,y) is largest at $(2,6)$ where it has value $4 + 36$.

12. (10 points)(Extra Credit) Consider the two functions
$f(x,y) = x^2 + 2y^2$ and $g(x,y) = 2x^2 + y^2$.
Start by holding $f(x,y) = 1$ constant and finding the Maximum value M for $g(x,y)$. Next hold $g(x,y) = M$ constant at the value you found and find the Maximum value for $f(x,y)$.
ANS: For both steps the Lagrange multiplier equation ensuring that the directioins of $\overrightarrow{\nabla} f$ and $\overrightarrow{\nabla} g$ are the same are needed and only the constraint changes. Compute: $\overrightarrow{\nabla} f = \langle 2x, 4y\rangle$ and $\overrightarrow{\nabla} g = \langle 4x, 2y\rangle$ so the equations $2x = \lambda 4x$ and $4y = \lambda 2y$ yields the three cases ($x = 0$ or $\lambda = \frac{1}{2}$) and ($y = 0$ or $\lambda = 2$) respectively. For the first step use the constraint $f = 1$ or $x^2 + 2y^2 - 1 = 0$ and consider the three cases: If $x = y = 0$ there is no solution. If $x = 0$ and $\lambda = 2$ then $y = \frac{\pm 1}{\sqrt{2}}$. If $y = 0$ and $\lambda = \frac{1}{2}$ then $x = \pm 1$. Among these four critical cases g is maximized if $x = \pm 1$ and has value $M = 2$. The second step is similar but uses the constraint $g = 2$ or $2x^2 + y^2 - 2 = 0$ and again consider the three cases: If $x = y = 0$ there is no solution. If $x = 0$ and $\lambda = 2$ then $y = \pm\sqrt{2}$. If $y = 0$ and $\lambda = \frac{1}{2}$ then $x = \pm 1$. Among these four critical cases f is maximized if $y = \pm\sqrt{2}$ and has value 4.