Math 21C Final Answers, March 22 Winter 2018

1. Determine whether the following series converge. Specify the convergence tests you use.

   (a) $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2+\sqrt{n}}$

   ANS: $0 < \sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2+\sqrt{n}} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges by the **p-series test** so by the **comparison test** the original series converges also.

   (b) $\sum_{n=1}^{\infty}(-1)^n \frac{\ln(n)}{n}$

   ANS: $\lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{n\to\infty} \frac{n^{-1}}{1} = 0$ by **L'Hôpital's rule** and so the series converges by the **alternating series test**.

2. For each of the following find an upper bound for the error resulting from estimating the infinite sum with just the first 5 terms.

   (a) $\sum_{n=1}^{\infty} \frac{2n}{(1+n^2)^2}$

   ANS: The **integral test error bound** gives a maximum error of $\int_5^{\infty} \frac{2x dx}{(1+x^2)^2} = \int_{26} \infty \frac{du}{u^2} = \frac{1}{26}$ using the substitution $u = 1 + x^2$.

   (b) $\sum_{n=1}^{\infty}(-1)^n \frac{3}{2^{2n}}$

   ANS: The **alternating series error bound** gives a maximum error of $\frac{3}{n^{12}}$. Alternatively the exact formula for **geometric series** gives an exact error of $\frac{3}{2^{12}} \frac{1}{1-\frac{1}{4}} = \frac{3}{2^{12}} \frac{4}{5}$.

3. Determine the values of x for which the following series converges. Be sure to check the end points of the interval. $\sum_{n=1}^{\infty} \sqrt{n} \frac{x^n}{5^n}$

   ANS: By the **ratio test** write $p = \lim_{n\to\infty} \frac{x\sqrt{n}}{5\sqrt{n+1}} = \frac{x}{5}$. If $|p| < 1$ or $-5 < x < 5$ the series converges (absolutely even) and if $|p| > 1$ or $|x| > 5$ it diverges. If $x = \pm 5$ then $\lim_{n\to\infty}(\pm\sqrt{n}\frac{x^n}{5^n})$ does not exist so the series diverges by **n-th term test**.

4. Find the first three nonzero terms of the Taylor series about $x = 0$ for the following function. $f(x) = \cos(2x) - x\sin(x)$

   ANS: $f(x) = (1 - \frac{1}{2}(2x)^2 + \frac{1}{24}(2x)^4 + \ldots) - x(x - \frac{1}{6}x^3 + \ldots) = 1 + (2 - 1)x^2 + (\frac{16}{24} + \frac{1}{6})x^4 + \ldots$.

5. Find the ground speed (magnitude of the velocity vector) of a fly if the wind is blowing the fly $4\frac{\text{mi}}{\text{hr}}$ northeast while the fly is flying $3\frac{\text{mi}}{\text{hr}}$ west.

   ANS: The velocity vector is $\frac{4}{\sqrt{2}}\langle 1, 1\rangle + \langle -3, 0\rangle$ so the speed is $\sqrt{(2\sqrt{2} - 3)^2 + 8}$.

6. Find an equation for the plane through the points $(1, 1, 1)$, $(1, 2, 3)$ and $(-1, 0, 3)$.

ANS: The cross product $(\langle 1,2,3\rangle - \langle 1,1,1\rangle) \times (\langle 1,2,3\rangle - \langle -1,0,3\rangle) = \langle -4,4,-2\rangle$ is perpendicular to the plane so an equation is $-4x+4y-2z = c$ and $(1,1,1)$ is on the plane so $c = -2$. (As a check the other two points also satisfy the equation.)

7. Consider the function $f(x,y) = 3\sqrt{4-x^2-y^2}$.

   (a) Find and sketch the domain of f.

   ANS: The disk of radius 2 centered at the origin.

   (b) Find and sketch the range of f.

   ANS: The interval $[0,6]$.

   (c) Describe the surface $z = f(x,y)$.

   ANS: The top half of an ellipsoid.

8. Consider again the surface $z = 3\sqrt{4-x^2-y^2}$. Find a parametric equation for the line normal to the surface at the point with $x = y = 1$.

   ANS: The surface satisfies $g(x,y,z) = x^2+y^2+\frac{z^2}{9}-4 = 0$ so the direction of the line is $\nabla g = \langle 2x, 2y, \frac{2}{9}z\rangle$ at the point $(1,1,3\sqrt{2})$. A parametric equation is therefore $\langle 1+2t, 1+2t, 3\sqrt{2}+\frac{2\sqrt{2}}{3}t\rangle$.

9. Laplace's equation for heat in a plate is satisfied by $f(x,y)$ if $f_{xx}+f_{yy} = 0$. Determine whether each of the following satisfy Laplace's equation.

   (a) $f(x,y) = e^{-2y}\cos(3x)$

   ANS: $f_{xx} = -9e^{-2y}\cos(3x)$ and $f_{yy} = 4e^{-2y}\cos(3x)$ so Laplace's equation is not satisfied.

   (b) $f(x,y) = \ln(x^2+y^2)$

   ANS: $f_{xx} = \frac{d}{dx}(\frac{2x}{x^2+y^2}) = \frac{2y^2-2x^2}{(x^2+y^2)^2}$ and similarly $f_{yy} = \frac{2x^2-2y^2}{(y^2+x^2)^2}$ so Laplace's equation is satisfied.

10. Find all the local maxima, local minima and saddle points of the function $f(x,y) = x^3 - y^3 - 2xy + 6$.

    ANS: Every such point is critical so use the **first derivative test** and find points (x,y) for which $0 = \nabla f = \langle 3x^2-2y, -3y^2-2x\rangle$. This requires that $x = -\frac{3}{2}y^2$ and $y = \frac{3}{2}x^2$ so $y = \frac{3}{2}(-\frac{3}{2}y^2)^2 = \frac{27}{8}y^4$ and either $y = 0$ so $x = 0$ or $y = \frac{2}{3}$ so $x = -\frac{2}{3}$. Thus there are two critical points $(0,0)$ and $(-\frac{2}{3}, \frac{2}{3})$. To determine their type use the **second derivative test** and compute $f_{xx} = 6x$, $f_{yy} = -6y$ and $f_{xy} = f_{yx} = -2$ so that $H = f_{xx}f_{yy} - f_{xy}f_{yx} = -36xy - 4$. At the point $(0,0)$ one has $H = -4 < 0$ so the point is a saddle point. At the point $(-\frac{2}{3}, \frac{2}{3})$ one has $H = 12 > 0$ so the point is a local extremum and further $f_{xx} = -4 < 0$ so the point is a local maximum.

11. Find the maximum value of the function $f(x, y, z) = x - 2y + 3z$ on the sphere $x^2 + y^2 + z^2 = 14$.

    ANS: This is a constrained optimization so use the **Lagrange multiplier** method with f as given and $g = x^2 + y^2 + z^2 - 14$ so that at the maximum one has $\nabla f = \langle 1, -2, 3\rangle = \lambda \nabla g = \lambda\langle 2x, 2y, 2z\rangle$. This means that $x = \frac{1}{2}\lambda^{-1}$, $y = -\lambda^{-1}$ and $z = \frac{3}{2}\lambda^{-1}$ so using $g = 0$ gives $(\frac{1}{4} + 1 + \frac{9}{4}\lambda^{-2} = \frac{14}{4})\lambda^{-2} = 14$ so $\lambda^{-1} = 2$ and the maximum value is $f(1, -2, 3) = 14$.

12. (Optional extra credit problem.) Evaluate the sum $\sum_{n=0}^{\infty} \frac{1}{(4n)!}$.

    ANS: Consider the Maclaurin series for $\frac{1}{4}e^x + \frac{1}{4}e^{-x} + \frac{1}{2}\cos(x) = \sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!}$ so the sum is $\frac{e}{4} + \frac{1}{4e} + \frac{\cos(1)}{2}$.