Math 21C Midterm II Friday, May 10 Spring 2024

Name:

ID:

You may use one page of notes but not a calculator or textbook.
Please do not simplify your answers.

1. (12 pts: Power Series)
Determine the x values for which the following power series converges:

$$\sum_{m=0}^{\infty} m\left(\frac{x}{3}-1\right)^m.$$

$$\lim_{n\to\infty}\left|\frac{(n+1)\left(\frac{x}{3}-1\right)^{n+1}}{n\left(\frac{x}{3}-1\right)^n}\right| = \left|\frac{x}{3}-1\right| \lim_{n\to\infty}\left|\frac{n+1}{n}\right| = \left|\frac{x}{3}-1\right| < 1$$

$$-1 < \frac{x}{3}-1 < 1 \Rightarrow 0 < \frac{x}{3} < 2 \Rightarrow 0 < x < 6$$

$x=0$,

$$\sum_{m=0}^{\infty} m(-1)^m$$

diverges by nth term

$x=6$

$$\sum_{m=0}^{\infty} m$$

diverges by nth term

$\Rightarrow$ Power series converges when

$$\boxed{x \in (0,6)}$$

2. (11 pts for each part: Taylor Polynomials)
Find the first three nonzero terms for the Taylor series associated to:

(a) $f(x) = \ln(x)$ about $x = 1$.

$f(1) = 0$

$f'(x) = \frac{1}{x}, f'(1) = 1$

$f''(x) = \frac{-1}{x^2}, f''(1) = -1$

$f'''(x) = \frac{2}{x^3}, f'''(1) = 2$

First three nonzero:

$$\boxed{\frac{1(x-1)}{1!} - \frac{1(x-1)^2}{2!} + \frac{2(x-3)^3}{3!}}$$

(b) $f(x) = e^x \cos(5x)$ about $x = 0$.

$e^x = \sum \frac{x^n}{n!}, \quad \cos(5x) = \sum \frac{(-1)^n (5x)^{2n}}{(2n)!}$

$\left(1 + x + \frac{x^2}{2}\right)\left(1 - \frac{25x^2}{2}\right)$

$= 1 - \frac{25x^2}{2} + x + \frac{x^2}{2}$

(Don't include cubic terms because we only see 3 single powers)

$$\boxed{= 1 + x - 12x^2}$$

3. (11 pts: Taylor Remainder)
Estimate the error if the Maclauren polynomial $P_1(x)$ associated to $f(x) = \sqrt{1-x}$ is used to estimate $\frac{1}{\sqrt{2}}$.
(Recall that the Maclauren polynomial is the Taylor polynomial about $x = 0$).

$f(x) = \sqrt{1-x} = \frac{1}{\sqrt{2}}, \Rightarrow x = \frac{1}{2}$

$R_1(\frac{1}{2}) \leq \left| M \frac{(\frac{1}{2})^2}{2} \right|$, $M \geq |f''(t)|$, t btwn $0, 1/2$

$f'(x) = \frac{-1}{2}(1-x)^{-1/2}$, $f''(x) = \frac{-1}{4}(1-x)^{-3/2}$

note: $f''(x) < 0$ for $x \in [0, 1/2]$

$f''(0) = \frac{-1}{4}$, $f''(1/2) = \frac{-1}{4}(1/2)^{-3/2} = -(\frac{1}{2})^{1/2}$

So, $M \geq \sqrt{1/2}$. Thus,

$$\boxed{R_1(1/2) \leq \left| \sqrt{1/2} \frac{(1/2)^2}{2} \right| = \left| \frac{1}{8\sqrt{2}} \right|}$$

4. (11 pts: Vectors)
Let $\mathbf{u} = \langle 1, 1, 1 \rangle$ and $\mathbf{v} = \langle 1, 2, 3 \rangle$ and write θ for an angle between them.

(a) Find $\sin(\theta)$.

$$\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = i(3-2) - j(3-1) + k(2-1)$$

$$= i - 2j + k, \quad |u \times v| = \sqrt{6}$$

$$\Longrightarrow \sin\theta = \frac{\sqrt{6}}{\sqrt{42}}$$

(b) Find $\cos(\theta)$.

$$\cos\theta = \frac{u \cdot v}{|u||v|} = \frac{1+2+3}{\sqrt{3}\sqrt{14}} = \boxed{\frac{6}{\sqrt{42}}}$$

5. (11 pts: Forces)
Consider a weight suspended by two wires. The force vector for one wire is $\overrightarrow{F}_1 = \langle 2, 1 \rangle$ Newtons. The direction vector of the second is $\overrightarrow{v} = \frac{1}{\sqrt{5}}\langle -1, 2 \rangle$ (so the second force vector $\overrightarrow{F}_2$ is a multiple of the vector $\overrightarrow{v}$).
Find the magnitude in Newtons of the weight that is suspended.

Equilibrium: $\vec{F}_1 + \vec{F}_2 + \vec{w} = 0$, so

First Components

$$2 + f_2\left(\frac{-1}{\sqrt{5}}\right) = 0$$

$$2 = \frac{f_2}{\sqrt{5}}$$

$$2\sqrt{5} = f_2$$

So

$$\vec{F}_2 = \langle -2, 4 \rangle$$

Second Components

Let w = downwards comp of weight $\vec{w}$.

$$1 + 4 + w = 0$$

So $w = -5$

$\boxed{\Rightarrow \text{magnitude of weight is } 5}$

6. (11 pts: Lines)
Find parametric equations for the line passing through the two points $P(1,3,5)$ and $Q(2,4,6)$.

Let $\vec{p} = \langle 1,3,5 \rangle$, and

$\vec{d} = \overrightarrow{PQ} = \langle 2-1, 4-3, 6-5 \rangle = \langle 1,1,1 \rangle$

So $\vec{r}(t) = \langle 1+t, 3+t, 5+t \rangle$

OR

$\vec{r}(t) = \langle 1,3,5 \rangle + t\langle 1,1,1 \rangle$

OR

$x = 1+t,\ y = 3+t,\ z = 5+t$

$\vec{PQ} = \langle 1, 1, 1 \rangle$, $\vec{PR} = \langle -1-1, 0-3, 1-5 \rangle = \langle -2, -3, -4 \rangle$

7. (11 pts: Planes) Consider the three points $P(1, 3, 5)$, $Q(2, 4, 6)$ and $R(-1, 0, 1)$.

(a) Find an equation for the plane containing P, Q and R.

$$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ -2 & -3 & -4 \end{vmatrix} = \langle -1, 2, -1 \rangle$$

So, $-x + 2y - z + d = 0$

$P = (1, 3, 5)$ is on plane, so

$-1 + 2(3) - 5 + d = 0$

$6 - 6 + d = 0 \Rightarrow d = 0$

$\boxed{\text{Plane is } -x + 2y - z = 0}$

(b) Find the area of the triangle with corners at P, Q and R.

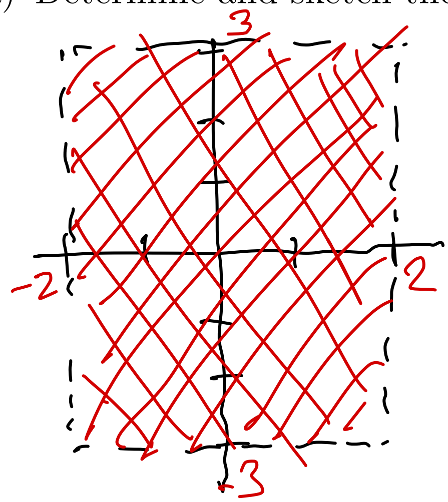
$$\text{Area } \triangle = \frac{|\vec{PQ} \times \vec{PR}|}{2}$$

$$\boxed{= \frac{\sqrt{6}}{2}}$$

8. (11 pts: Functions)
Consider the function $f(x, y) = \sqrt{4 - x^2} + \sqrt{9 - y^2}$.

(a) Determine and sketch the domain of f in the plane.

Domain:
$4 - x^2 \geq 0$
$|x| \leq 2$
$9 - y^2 \geq 0$
$|y| \leq 3$



(b) Determine the range of f.

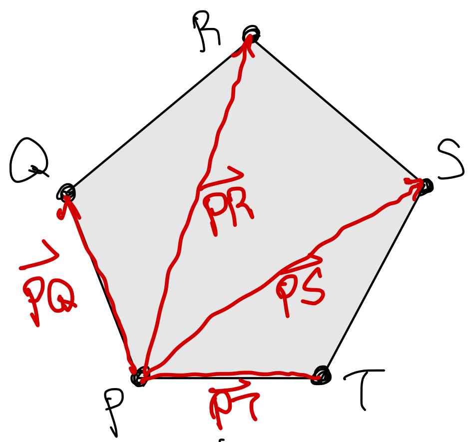
Range: $\sqrt{\cdot} \not< 0$, so min is 0. maximum achieved at $(x, y) = (0, 0)$, $f(0,0) = 5$

$\sqrt{4 - x^2}$ halfcircle, max at $x = 0$
$\sqrt{9 - y^2}$ halfcircle, max at $y = 0$

$Ran(f) = [0, 5]$

9. (10 pts: Extra Credit... you may skip this problem)
Consider the five points:
$P(2, 0, 0)$, $Q(0, 4, 0)$, $R(0, 2, 1)$, $S(1, -2, 2)$ and $T(2, -4, 2)$.

Find the area of the pentagon with edges:
$\overline{PQ}$, $\overline{QR}$, $\overline{RS}$, $\overline{ST}$ and $\overline{TP}$.



$\vec{PQ} = \langle -2, 4, 0 \rangle$
$\vec{PR} = \langle -2, 2, 1 \rangle$
$\vec{PS} = \langle -1, -2, 2 \rangle$
$\vec{PT} = \langle 0, -4, 2 \rangle$

$$\left\| \begin{vmatrix} i & j & k \\ -2 & 4 & 0 \\ -2 & 2 & 1 \end{vmatrix} \right\| = \left| i(4) - j(-2) + k(-4+8) \right| = \left| \langle 4, 2, 4 \rangle \right| = \sqrt{36} = 6$$

So Area$(\triangle PQR) = 3$

$$\left\| \begin{vmatrix} i & j & k \\ -2 & 2 & 1 \\ -1 & -2 & 2 \end{vmatrix} \right\| = \left| i(4+2) - j(-4+1) + k(4+2) \right| = \left| \langle 6, 3, 6 \rangle \right| = \sqrt{81} = 9$$

, so Area$(\triangle PRS) = 9/2$

$$\left\| \begin{vmatrix} i & j & k \\ -1 & -2 & 2 \\ 0 & -4 & 2 \end{vmatrix} \right\| = \left| i(-4+8) - j(-2) + k(4) \right| = \left| \langle 4, 2, 4 \rangle \right| = \sqrt{36} = 6$$

so Area$(\triangle PST) = 3$

$$\boxed{\text{Thus, Area}(⬠ PQRST) = 3 + 3 + 9/2 = 21/2}$$