Math 21C Midterm III Solutions Spring 2024 You may use one page of notes but not a calculator or textbook.
Please do not simplify your answers.

1. (16 points: Partial Derivatives) Consider the function $f(x, y) = x^4 + x^3y + x^2y^2 + xy^3 + y^4$. Compute the following partial derivatives:

   (a) f_x
   ANS: $f_x = 4x^3 + 3x^2y + 2xy^2 + y^3$.

   (b) f_{xy}
   ANS: $f_{xy} = 3x^2 + 4xy + 3y^2$.

   (c) f_{yxx}
   ANS: $f_{yxx} = f_{xyx} = 6x + 4y$.

   (d) f_{xyyx}
   ANS: $f_{xyyx} = f_{yxxy} = 4$.

2. (18 points: Gradient Vector)
   Consider the function $f(x, y) = x^2 - 3xy + y^2$.

   (a) Find the gradient vector $\overrightarrow{\nabla f}(1, 2)$.
   ANS: $\overrightarrow{\nabla f}(1, 2) = \langle 2x - 3y, -3x + 2y\rangle|_{(x,y)=(1,2)} = \langle -4, 1\rangle$.

   (b) Find the directional derivative $(D_{\overrightarrow{u}}f)(1, 2)$ in the direction of the unit vector $\overrightarrow{u} = \langle \frac{3}{5}, \frac{4}{5}\rangle$.
   ANS: $D_{\overrightarrow{u}}f(1, 2) = \overrightarrow{\nabla f}(1, 2) \cdot \langle \frac{3}{5}, \frac{4}{5}\rangle = \langle -4, 1\rangle \cdot \langle \frac{3}{5} \frac{4}{5}\rangle = -\frac{8}{5}$.

   (c) Find the largest possible value for the directional derivative $(D_{\overrightarrow{v}}f)(1, 2)$ in the direction of any unit vector $\overrightarrow{v}$.
   ANS: $|\overrightarrow{\nabla f}(1, 2)| = |\langle -4, 1\rangle| = \sqrt{17}$.

3. (15 points: Extant Limit) Evaluate the following limit: $\lim_{(x,y)\to(2,2)} \frac{x^2-xy-x+y}{x^2-y^2}$.
ANS: $\lim_{(x,y)\to(2,2)} \frac{x^2-xy-x+y}{x^2-y^2} = \lim_{(x,y)\to(2,2)} \frac{(x-y)(x-1)}{(x-y)(x+y)} = \lim_{(x,y)\to(2,2)} \frac{x-1}{x+y} = \frac{1}{4}$.

4. (15 points: Nonexistent Limit)
Verify that the following limit does not exist: $\lim_{(x,y)\to(0,0)} \frac{xy(x+y)}{x^3+y^3}$.
ANS: Along the line $y = 0$, $\lim_{(x,0)\to(0,0)} \frac{0}{x^3+0^3} = 0$. Along the line $y = x$, $\lim_{(x,x)\to(0,0)} \frac{xx(x+x)}{x^3+x^3} = \frac{2}{2} = 1 \neq 0$. Since the limits along these two lines are different the limit does not exist.

5. (16 points: Chain Rule)
Consider functions with $z = f(x, y)$, $x = u(s, t)$ and $y = v(s, t)$.

   (a) Write a chain rule formula for $\frac{\partial z}{\partial t}$. (A branch diagram might help.)
   ANS: $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$.

   (b) Two of the functions are $z = f(x, y) = x^2 + y^2$ and $x = u(s, t) = e^{2s+3t}$. Consider the point $s = t = 0$. At this point $y = v(0, 0) = 3$ and $y_t = \frac{\partial v}{\partial t}(0, 0) = 7$.
   Find $z_t = \frac{\partial z}{\partial t}(0, 0)$ at the point $s = t = 0$.
   ANS: At this point $x = e^0 = 1$, $\frac{\partial z}{\partial x} = 2x = 2$, $\frac{\partial z}{\partial y} = 2y = 6$ and $\frac{\partial x}{\partial t} = 3e^{2s+3t} = 3$ so by the previous part $\frac{\partial z}{\partial t} = (2)(3) + (6)(7) = 48$.

6. (20 points: Critical Points) Consider the graph of the function $f(x, y) = 2x^2 - 2x^2y + y^2$.

   (a) Find the three critical points for this graph.
   ANS: $\overrightarrow{\nabla f} = \langle 4x - 4xy, -2x^2 + 2y\rangle$ which is $\langle 0, 0\rangle$ if $f_x = f_y = 0$. To solve this system of equations use the second to get $y = x^2$ and substitute into the first to get $0 = 4x - 4x^3 = 4x(1 - x)(1 + x)$. Thus the critical points are $\langle 0, 0\rangle$, $\langle 1, 1\rangle$ and $\langle -1, 1\rangle$.

   (b) Use the second derivative test to identify which (if any) of these critical points are saddle points.
   ANS: $f_{xx} = 4 - 4y$, $f_{xy} = f_{yx} = -4x$ and $f_{yy} = 2$. Thus among the three critical points $f_{xx}f_{yy} - f_{xy}^2 < 0$ only at $\langle 1, 1\rangle$ and $\langle -1, 1\rangle$ which are hence the two saddle points.

7. (10 points: Extra Credit... you may skip this problem)
   Consider the function $f(x, y) = \frac{xy}{\sqrt{x^2+y^2}}$.
   Determine whether the following limit exists: $\lim_{(x,y)\to(0,0)} |\overrightarrow{\nabla f}(x, y)|^2$.
   ANS: $\overrightarrow{\nabla f}(x, y) = (x^2 + y^2)^{\frac{-3}{2}}\langle y^3, x^3\rangle$ so $|\overrightarrow{\nabla f}(x, y)| = \sqrt{\frac{x^6+y^6}{(x^2+y^2)^3}}$. Along the line $y = 0$ the limit is 1 but along the line $y = x$ the limit is $\frac{1}{2}$ so the limit does not exist. As an asside note that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$ so the function is continuous but not differentiable at the origin.