Math 21C Practice Midterm III Solutions Spring 2024

You may not use a calculator.
You may use one page of notes.
You may not use the textbook.
Please do not simplify answers.

1. (15 pts: Partial Derivatives)
   Consider $z = 3x + \ln(x^2 + y)$. Compute the partial derivatives:

   (a) z_x

   (b) z_y

   (c) z_{xx}

   ANS:

   (a) $z_x = 3 + \frac{2x}{x^2+y}$

   (b) $z_y = \frac{1}{x^2+y}$

   (c) $z_{xx} = \frac{2(x^2+y)-(2x)(2x)}{(x^2+y)^2}$

2. (21 pts: Differential)
   Consider the surface given by $z = xy + y^3 - x^2$ and the point $P(2, 1)$.

   (a) What is the maximum possible value of $D_{\overrightarrow{u}} f$ at P and for this maximum in what direction is $\overrightarrow{u}$?

   (b) Find a point Q so that the directional derivative at $P(2, 1)$ in direction $\overrightarrow{PQ}$ is zero.

   (c) Find an equation for the tangent plane to the surface at the point $(2, 1)$.

ANS: The surface is the graph of the function $f(x, y) = xy + y^3 - x^2$. Start with the gradient vector for f at P. $\overrightarrow{f} = \langle f_x, f_y \rangle = \langle y - 2x, x + 3y^2 \rangle$ so $\overrightarrow{f}(2, 1) = \langle -3, 5 \rangle$.

(a) The maximum directional derivative of f at P will occur in the direction of $\overrightarrow{f}(2, 1) = \langle -3, 5 \rangle$ which is $\overrightarrow{u} = \frac{1}{\sqrt{9+25}}\langle -3, 5 \rangle$. The maximum possible value is $D_{\overrightarrow{u}} f(2, 1) = \overrightarrow{u} \cdot \overrightarrow{f}(2, 1) = \frac{1}{\sqrt{9+25}}\langle -3, 5 \rangle \cdot \langle -3, 5 \rangle = \frac{9+25}{\sqrt{9+25}}$.

(b) The directional derivative $D_{\overrightarrow{v}} f(2, 1) = 0$ if $\overrightarrow{v}$ is perpendicular to $\overrightarrow{\nabla f}(2, 1) = \langle -3, 5 \rangle$ such as the direction of $\langle 5, 3 \rangle$ so choose $Q(7, 4)$ so that $\overrightarrow{PQ} = \langle 5, 3 \rangle$. (Any other point except P on the line containing P and Q would also work for Q.)

(c) $f(2, 1) = 2 + 8 - 4 = 6$ so the plane is given by $z = 6 - 3(x - 2) + 5(y - 1)$.

3. (14 pts: Extant Limit)
Evaluate the following limit:

$$\lim_{(x,y)\to(1,-1)} \frac{1 - \sqrt{x + y + 1}}{x + y}.$$

ANS: The function is not defined at $(1, -1)$ so direct substitution fails. Try multiplication by conjugates and simplification $\frac{1-\sqrt{x+y+1}}{x+y} = \frac{(1+\sqrt{x+y+1})(1-\sqrt{x+y+1})}{(1+\sqrt{x+y+1})(x+y)} = \frac{1-(x+y+1)}{(1+\sqrt{x+y+1})(x+y)} = -\frac{x+y}{(1+\sqrt{x+y+1})(x+y)} = \frac{-1}{(1+\sqrt{x+y+1})}$ on the domain of the given function and hence the limit is $\frac{-1}{(1+\sqrt{1-1+1})} = \frac{-1}{2}$.

4. (14 pts: Nonexistant Limit)

Verify that the following limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^4+y^4}.$$

ANS: Use the two line method. If $y=0$ the imit is $\lim_{x\to 0}\frac{0}{x^4+0} = 0$. If $y=x$ the imit is $\lim_{x\to 0}\frac{x^4}{x^4+x^4} = \frac{1}{2}$. These are not equal so the limit does not exist.

5. (15 pts: Chain Rule)
Consider $z = f(x,y)$, $x = 3r+s^2$ and $y = rs$.
Use the chain rule to find the second-order partial derivative $\frac{\partial^2 z}{\partial s^2}$.
Express your answer in terms of r, s and partial derivatives of z with respect to x and y.

ANS: $\frac{\partial^2 z}{\partial s^2} = z_{ss} = (z_s)_s = (z_x x_s + z_y y_s)_s = (2sz_x + rz_y)_s = 2z_x + 2s((z_x)_s) + r((z_y)_s) = 2z_x + 2s(z_{xx}x_s + z_{xy}y_s) + r(z_{yx}x_s + z_{yy}y_s) = 2z_x + (2s)(2s)z_{xx} + (2s)(r)z_{yx} + (r)(2s)z_{xy} + (r)(r)z_{yy}$.

6. (21 pts: Critical Points)
Consider the function $f(x,y) = x^3 - 3xy + y^3$.
   (a) Find all its local maxima.
   (b) Find all its local minima.
   (c) Find all its saddle points.

ANS: First find the critical points for f using the first derivatives $f_x = 3x^2 - 3y$ and $f_y = -3x + 3y^2$. Solve for $0 = f_x = f_y$. Use $0 = f_x = 3(x^2 - y)$ to get $y = x^2$. Then use $0 = f_y = -3x + 3y^2 = -3x + 3(x^2)^2 = 3(x^4 - x) = 3x(x-1)(x^2+x+1)$ to get $x = 0$ or $x = 1$. Hence the two critical points are $(0,0)$ and $(1,1)$.

Next use the second derivative test to identify the type of each critical point. This requires the second deriatives $f_{xx} = 6x$, $f_{xy} = -3 = f_{yx}$ and $f_{yy} = 6y$.

(a) There are no local Maxima.

(b) At $(1, 1)$ the discriminant is $f_{xx}f_{yy} - f_{xy}^2 = 36 - 9 > 0$ and $f_{xx} > 0$ so there is a local minimum.

(c) At $(0, 0)$ the discriminant is $f_{xx}f_{yy} - f_{xy}^2 = -9 < 0$ so there is a saddle point.

7. (10 pts: Extra Credit... you may skip this problem)