Math 108 Spring 2020 Final Exam
Due Wednesday June 10 at Midnight

1. (15 points) Determine the three pairs of equivalent sentences below and find their truth tables:
   (a) P
   (b) $P \wedge (\sim Q)$
   (c) $\sim ((\sim P) \wedge Q)$
   (d) $\sim [P \implies (P \wedge Q)]$
   (e) $P \vee [(\sim Q) \vee (P \vee (\sim Q))]$
   (f) $P \iff [(Q \implies P) \vee ((\sim Q) \implies P)]$

2. (15 points) (Induction) In the universe $\mathbb{N}$ prove that
$$(\forall n) \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

3. (20 points) Consider three possible universes: $\mathbb{N}, \mathbb{Z}$ and $\mathbb{R}$.
   (a) Determine for each of the following four sentences and each of the three above universes whether the sentence is true in the universe.
      i. $(\forall x)(\exists! y)\ x^3 = y^2$
      ii. $(\forall x)(\exists! y)\ x^2 = y^3$
      iii. $(\exists! y)(\forall x)\ xy^2 = y$
      iv. $(\exists! y)(\forall x)\ xy^2 = x$
   (b) Prove one case in which the sentence is true.
   (c) Prove one case in which the sentence is false.

4. (15 points) Prove that if n is a natural number then n is a multiple of three iff $n^2 - 1$ is not a multiple of three.

5. (15 points) Consider the relation S from $\mathbb{R}$ to $\mathbb{R}$ given by xSy if $x - y \in \mathbb{Z}$.
   (a) Show that S is an equivalence relation.
   (b) Find three different real numbers a, b and c for which $\overline{a} = \overline{b} \neq \overline{c}$.

6. (20 points) If A is a set consider the relation
$R = \{((x, y), \{x, y\}) | (x \in A) \wedge (y \in A)\}$ from $A \times A$ to $\mathcal{P}A$.
   (a) Draw an arrow diagram (eg Fig 3.1.1.b) for R if $A = \{1, 2, 3\}$.
   (b) Show that for any set A the relation R is a function.
   (c) Show that for any set A the relation R is not onto.
   (d) Show that for any set A wth at least two elements the relation R is not one-to-one.