Math 108 Winter 2021 Final Exam
Due Friday March 19 at Noon

Feel free to work on the exam for the full 48 hours that it is available and make use of the materials (texts, notes, lectures) from the course. Do not discuss the problems with others or make use of other assistance. Please write and sign the course honor code on the exam:
I pledge on my honor that I have not given or received any unauthorized assistance on this examination.

1. (14 points)(Logic) Consider the English sentence:
If it is noon then if I am hungry it is raining, and it is either noon or raining.
   (a) Find a sentence in first order logic representing this English sentence.
   (b) Find another equivalent sentence in first order logic which does not involve $\implies$.
   (c) Find the truth table for the sentences you found.

2. (14 points)(Quantifiers) Consider the logical sentence in the universe of discourse $\mathbb{N}$:
$$(\exists n)(\forall r)(\exists k)[(k < n) \wedge (k > r)].$$
   (a) Write a negation of this sentence which does not involve $\sim$.
   (b) Prove either the sentence or its negation (whichever is true).

3. (15 points)(Sets) If A and B are sets contained in a universe (of discourse) U define the following four sets:
$W = (A - B) \cup (B - A)$,
$X = (A - B) \cup (A \cap B)$,
$Y = (A \cup B) - (A \cap B)$ and
$Z = \{x \in U | (\exists!\ S \in \{A, B\})\ x \in S\}$.
   (a) Show that for every universe U and pair of sets A and B three of the sets W, X, Y and Z are equal.
   (b) Show that there is a universe U and a pair of sets A and B for which one of the sets W, X, Y and Z is not equal to the other three.

4. (14 points)(Induction) Show that $(\forall n \in \mathbb{N})$
$$3\sum_{r=1}^{n}(r-1)r = (n-1)n(n+1).$$

5. (14 points)(Relations) Show that on the set $\mathbb{N}_3$ there is a unique equivalence relation which contains $(1, 2)$ but not $(2, 3)$.

6. (15 points)(Bijection) Given a universe U, consider the set $A = \{(w, x) \in U \times U | w \leq x\}$ and the relation $R = \{((w, x), (x, y)) \in A \times A | y - x = x - w\}$ on A.
   (a) If $U = \mathbb{Z}$ find all elements of R with $(3, 5) \in A$ as one entry (either first or second).
   (b) Show that if $U = \mathbb{Z}$ then R is a bijective function.
   (c) Show that if $U = \mathbb{N}$ then R is not a bijective function.

7. (14 points)(Proofs) Consider the false claim:
If $n \in \mathbb{N}$, $\overline{x}, \overline{y} \in \mathbb{Z}_n$ and either $\overline{x} - \overline{y} \neq \overline{0}$ or $\overline{x} + \overline{y} \neq \overline{0}$ then $\overline{x} \neq \overline{y}$.
Clearly identify a serious flaw in each of the following attempts to prove this claim.
   (a) "Proof:" If it is not true that either $\overline{x} - \overline{y} \neq \overline{0}$ or $\overline{x} + \overline{y} \neq \overline{0}$ then both $\overline{x} - \overline{y} = \overline{0}$ and $\overline{x} + \overline{y} = \overline{0}$. The first equation implies that $\overline{x} = \overline{y}$. Therefore the claim is true. q.e.d.
   (b) "Proof:" If $n \in \mathbb{N}$ and $\overline{x} = \overline{0}$ and $\overline{y} = \overline{1}$ then $\overline{x} + \overline{y} = \overline{1} \neq \overline{0}$ so the hypothesis is true. Also $\overline{x} \neq \overline{y}$ so the conclusion is also true. Therefore the claim is true. q.e.d.
   (c) "Proof:" To show the contrapositive holds assume that $\overline{x} = \overline{y}$. Hence $\overline{x} - \overline{y} = \overline{0}$ so either $\overline{x} - \overline{y} = \overline{0}$ or $\overline{x} + \overline{y} = \overline{0}$. Therefore the claim is true. q.e.d.