Math 108 Spring 2021 Practice Final Problems

1. Decide whether the following is a tautology:
   $[(\sim Q) \wedge (P \implies Q)] \implies P$.

2. Prove that $(\forall n \in \mathbb{N})(5n+1 \text{ is even}) \implies (2n^2+3n+4 \text{ is odd})$.

3. Prove that in the universe $\mathbb{N}$ we have that
   a divides b
   iff $(\exists c)$ ac divides bc
   iff $(\forall c)$ ac divides bc.

4. Prove or find a counterexample: In any universe of sets
   $(\forall A, B, C, D)(C \subseteq A) \wedge (D \subseteq B) \implies (C \cap D) \subseteq (A \cap B)$.

5. Prove by induction that
   $$(\forall n \in \mathbb{N}) \quad \sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}.$$

6. Prove by induction and using Thm 2.6.4 that if $\mathcal{A} = \{A_r : r \in \mathbb{N}, r \leq n\}$ is a finite family of finite sets then $\overline{\overline{A_1 \times A_2 \times \ldots \times A_n}} = \overline{\overline{A_1}} \cdot \overline{\overline{A_2}} \cdot \ldots \cdot \overline{\overline{A_n}}$.

7. Find a number a so that $h \cup g$ is a function from $\mathbb{R}$ to $\mathbb{R}$ if $h(x) = |x+1|$ is a function from $(-\infty, 1]$ to $\mathbb{R}$ and $g(x) = a - |x-3|$ is a function from $[0, \infty)$ to $\mathbb{R}$.

8. Show that there is a function $g = f^{-1}$ from $\mathbb{R} - \{5\}$ to $\mathbb{R} - \{3\}$ if $f(x) = \frac{5(x-1)}{x-3}$.

9. Fix a universe of sets. Define a relation
   $R = \{(A, B) | [(\exists f)(f : A \to B \text{ is onto})] \wedge [(\exists g)(g : B \to A \text{ is onto})]\}$.
   Prove that R is an equivalence relation.

10. Prove that in the universe $\mathbb{R}$ we have that
    $\sim (\exists x > 0)(\forall y > 0)(\exists n)(\forall m > n)(\frac{1}{m} - x \leq y) \wedge (x - \frac{1}{m} \leq y)$.

11. Determine with a short explaination which of the following have cardinality $\overline{\overline{\mathbb{N}_k}}$, $\overline{\overline{\mathbb{N}}}$, $\overline{\overline{(0,1)}}$ or none of these.

    (a) $\{(x, y) \in \mathbb{R}^2 | x = y^2\}$

    (b) $\mathbb{Q} \times \mathbb{N}$

    (c) $\mathcal{P}(\mathbb{R})$

    (d) $\mathbb{Z}/R$ if R is the equivalence relation $R = \{(x, y) \subseteq \mathbb{Z}^2 | x^2 = y^2\}$

    (e) $\mathbb{Z}/R$ if R is the equivalence relation $R = \{(x, y) \subseteq \mathbb{Z}^2 | 5 \text{ divides } x - y\}$