Math 108 Spring 2020 Practice Final Answer Sketch

1. Decide whether the following is a tautology:
$[(\sim Q) \wedge (P \implies Q)] \implies P$.
ANS: It is not. The proposition is false if Q is true and P is false.

2. Prove that $(\forall n \in \mathbb{N})(5n+1 \text{ is even}) \implies (2n^2+3n+4 \text{ is odd})$.
ANS: Proof Sketch: Suppose n is a natural number and $5n+1$ is odd. Hence $2n^2+4$ is even and $5n$ is odd so n is odd and $3n$is odd. Hence $2n^2 3n+2$ is odd. *QED*

3. Prove that in the universe $\mathbb{N}$ we have that
a divides b
iff $(\exists c)$ ac divides bc
iff $(\forall c)$ ac divides bc.
ANS: Note that to prove A iff B iff C it sffices to prove C implies B, B implies A and A implies C. Note that in any nonempty universe $(\forall c)P(c)$ implies $(\exists c)P(c)$. Combining these it suffices to prove two things:
1) $[(\exists c)$ ac divides $bc]$ implies $[a$ divides $b]$
2) $[a$ divides $b]$ implies $[(\forall c)$ ac divides $bc]$
For 1): Suppose ac divides bc so $bc = nac$ for some integer n and bc is nonzero so $b = na$ and a divides b.
For 2): Suppose a divides b and c is a natural number. Hence $b = an$ so $bc = acn$ for some integer n and ac divides bc. *QED*

4. Prove or find a counterexample: In any universe of sets
$(\forall A, B, C, D)(C \subseteq A) \wedge (D \subseteq B) \implies (C \cap D) \subseteq (A \cap B)$.
ANS: This is true. Proof Sketch: Suppose A, B, C, D are sets with $(C \subseteq A)$, $(D \subseteq B)$ and $x \in C \cap D$. Hence $x \in C$ and $x \in D$ so $x \in A$ and $x \in B$. Hence $x \in A \cap B$. *QED*

5. Prove by induction that
$$(\forall n \in \mathbb{N}) \quad \sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}.$$
ANS: Write $P(n)$ for the given inequality at n.
For the basis step consider $P(1)$ which is $\frac{1}{1^2} \leq 2 - \frac{1}{1}$ and is true.
For the induction step assume that n is at least 2 and $P(n-1)$ is true. Hence $(\sum_{k=1}^{n-1}) + \frac{1}{n^2} \leq 2 - \frac{1}{n-1} + \frac{1}{n^2} = 2 - \frac{n^2-n+1}{(n-1)n^2} \leq 2 - \frac{n^2-n}{(n-1)n^2} = 2 - \frac{1}{n}$.
Hence $P(n)$ is true and by PMI: *QED*

6. Prove by induction and using Thm 2.6.4 that if $\mathcal{A} = \{A_r : r \in \mathbb{N}, r \leq n\}$ is a finite family of finite sets then $\overline{\overline{A_1 \times A_2 \times \ldots \times A_n}} = \overline{\overline{A_1}} \cdot \overline{\overline{A_2}} \cdot \ldots \cdot \overline{\overline{A_n}}$.
ANS: Write $P(n)$ for the assertion in which the family has n sets in it.
For the basis steps consider $P(1)$ which is the proposition that if A_1 is a

finite set then $\overline{\overline{A_1}} = \overline{\overline{A_1}}$. This is true. Consider $P(2)$ which is Thm 2.6.4 and hence true.
For the induction step assume that n is at least 3 and $P(n-1)$ and $P(2)$ are true. Assume that $\mathcal{A} = \{A_i | 1 \le i \le n\}$ is a family of n finite sets. By $P(n-1)$ we know that $\overline{\overline{A_1 \times A_2 \times \ldots \times A_{n-1}}} = \overline{\overline{A_1}} \cdot \overline{\overline{A_2}} \cdot \ldots \cdot \overline{\overline{A_{n-1}}}$. By $P(2)$ we know that $\overline{\overline{(A_1 \times A_2 \times \ldots \times A_{n-1}) \times A_n}} = \overline{\overline{A_1}} \cdot \overline{\overline{A_2}} \cdot \ldots \cdot \overline{\overline{A_n}}$. Finally $g : (A_1 \times A_2 \times \ldots \times A_{n-1}) \times A_n \to A_1 \times A_2 \times \ldots \times A_n$ given by $g(((a_1, \ldots a_{n-1}), a_n)) = (a_1, \ldots a_{n-1}, a_n)$ is a bijection. Hence by PCI: *QED*

7. Find a number a so that $h \cup g$ is a function from $\mathbb{R}$ to $\mathbb{R}$ if $h(x) = |x+1|$ is a function from $(-\infty, 1]$ to $\mathbb{R}$ and $g(x) = a - |x-3|$ is a function from $[0, \infty)$ to $\mathbb{R}$.
ANS: $a = 4$ is the only number that works.

8. Show that there is a function $g = f^{-1}$ from $\mathbb{R} - \{5\}$ to $\mathbb{R} - \{3\}$ if $f(x) = \frac{5(x-1)}{x-3}$.
ANS: Consider the function $g(y) = \frac{10}{y-5} + 3$. Compute that $f \circ g$ and $g \circ f$ are identity functions on $\mathbb{R} - \{5\}$ and $\mathbb{R} - \{3\}$ respectively. If $x \ne 5$ then $[f \circ g](x) = f(g(x)) = \frac{5(\frac{10}{y-5}+3-1)}{\frac{10}{y-5}-3+3} = \frac{5(10+2(y-5))}{10} = y$. If $y \ne 3$ then $[g \circ f](y) = \frac{10}{\frac{5(x-1)}{x-3}-5} + 3 = \frac{10(x-3)}{5(x-1)-5(x-3)} + 3 = x$. *QED*

9. Fix a universe of sets. Define a relation
$R = \{(A, B) | [(\exists f)(f : A \to B \text{ is onto})] \wedge [(\exists g)(g : B \to A \text{ is onto})]\}$.
Prove that R is an equivalence relation.
ANS: There are three things to check:
(reflexive) If A is a set then $I_A : A \to A$ is onto so ARA.
(symmetric) If A and B are sets with ARB then there is $f : A \to B$ which is onto and $g : B \to A$ which is onto so BRA.
(transitive) If A, B and C are sets with ARB and BRC then there are $f : A \to B$ and $h : B \to C$ both onto so $h \circ f : A \to C$ is onto. There are also $g : B \to A$ and $k : C \to B$ which are onto so $g \circ k : C \to A$ is onto. *QED*

10. Prove that in the universe $\mathbb{R}$ we have that
$\sim (\exists y > 0)(\forall x > 0)(\exists n)(\forall m > n)(\frac{1}{m} - x \le y) \wedge (x - \frac{1}{m} \le y)$.
ANS: This is equivalent to
$(\forall x > 0)(\exists y > 0)(\forall n)(\exists m > n)(\frac{1}{m} - x > y) \vee (x - \frac{1}{m} > y)$.
Suppose $x > 0$. Choose $y = \frac{x}{2}$. Suppose n is a natural number. Choose $m > n$ with $m > \frac{2}{x}$. Hence $\frac{1}{m} < \frac{x}{2}$ and $x - \frac{1}{m} > \frac{x}{2} = y$. *QED*

11. Determine with a short explaination which of the following have cardinality $\overline{\overline{\mathbb{N}_k}}$, $\overline{\overline{\mathbb{N}}}$, $\overline{\overline{(0,1)}}$ or none of these.

    (a) $A = \{(x, y) \in \mathbb{R}^2 | x = y^2\}$

ANS: $\overline{\overline{A}} = \overline{\overline{(0,1)}} = \overline{\overline{\mathbb{R}}}$ since the map $f : B \to \mathbb{R}$ with $f((x,y)) = x$ is a bijection.

(b) $B = \mathbb{Q} \times \mathbb{N}$

ANS: $\overline{\overline{B}} = \overline{\overline{\mathbb{N}}}$ since there is a bijection from $\mathbb{Q}$ to $\mathbb{N}$ and from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.

(c) $C = \mathcal{P}(\mathbb{R})$

ANS: $\overline{\overline{C}}$ is none of the above since there is no surjective map from $\mathbb{R}$ to $\mathcal{P}(\mathbb{R})$.

(d) $D = \mathbb{Z}/R$ if R is the equivalence relation $R = \{(x,y) \subseteq \mathbb{Z}^2 | x^2 = y^2\}$

ANS: $\overline{\overline{D}} = \overline{\overline{N}}$ since the map $f(\overline{x}) = |x| + 1$ is a bijection from D to $\mathbb{N}$.

(e) $E = \mathbb{Z}/R$ if R is the equivalence relation $R = \{(x,y) \subseteq \mathbb{Z}^2 | 5 \text{ divides } x - y\}$

ANS: $\overline{\overline{E}} = \overline{\overline{\mathbb{N}_5}}$ since $E = \{\overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$.