Math 108 Spring 2020 Final Exam
Due Wednesday June 10 at Midnight

1. (15 points) Determine the three pairs of equivalent sentences below and find their truth tables:

   (a) P

   (b) $P \wedge (\sim Q)$

   (c) $\sim ((\sim P) \wedge Q)$

   (d) $\sim [P \implies (P \wedge Q)]$

   (e) $P \vee [(\sim Q) \vee (P \vee (\sim Q))]$

   (f) $P \iff [(Q \implies P) \vee ((\sim Q) \implies P)]$

   ANS: (b) and (d) are both F unless P is T and Q is F.
   (c) and (e) are both T unless P if F and Q is T.
   (a) and (f) are equivalent.

2. (15 points) (Induction) In the universe $\mathbb{N}$ prove that

$$(\forall n) \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

   ANS: For the base case if $n = 1$ the left side is $LHS(1) = 1^2$ and the right side is $RHS(1) = 6/6$ which are equal. For the induction step if $LHS(n) = RHS(n)$ then $LHS(n+1) = LHS(n) + (n+1)^2 = RHS(n) + (n+1)^2 = (n+1)[\frac{2n^2+n}{6} + (n+1)] = (n+1)[\frac{2n^2+7n+6}{6}] = (n+1)[\frac{((n+1)+1)(2(n+1)+1)}{6}] = RHS(n+1)$. q.e.d.

3. (20 points) Consider three possible universes: $\mathbb{N}, \mathbb{Z}$ and $\mathbb{R}$.

   (a) Determine for each of the following four sentences and each of the three above universes whether the sentence is true in the universe.

      i. $(\forall x)(\exists! y) \; x^3 = y^2$
      ANS: F, F and F.

      ii. $(\forall x)(\exists! y) \; x^2 = y^3$
      ANS: F, F and T.

      iii. $(\exists! y)(\forall x) \; xy^2 = y$
      ANS: F, T and T.

      iv. $(\exists! y)(\forall x) \; xy^2 = x$
      ANS: T, F and F.

   (b) Prove one case in which the sentence is true.
   ANS: For (iv) with the universe $\mathbb{N}$ if $y = 1$ and with any x the equation holds (existence). If $x = 1$ and the equation holds then $y^2 = 1$ so $y = 1$ (uniqueness). q.e.d.

(c) Prove one case in which the sentence is false.
ANS: For (i) with the universe $\mathbb{R}$ and $x = 1$ both $y = 1$ and $y = -1$ satisfy the equation. q.e.d.

4. (15 points) Prove that if n is a natural number then n is a multiple of three iff $n^2 - 1$ is not a multiple of three.
ANS: For the forward implication assume for contradiction that $n = 3k$ and $n^2 - 1 = 3r$. Hence 1 is not a multiple of three and $1 = n^2 - (n^2 - 1) = 9k^2 - 3r = 3(3k^3 - r)$ is a multiple of three, a contradiction.
For the (contrapositive of) the backward implication assume that n is not a multiple of three and hence either $n = 3k + 1$ or $n = 3r + 2$. In the first case $n^2 - 1 = 9k^2 + 6k + 1 - 1 = 3(3k^2 + 2k)$ is a multiple of three. In the second case $n^2 - 1 = 9r^2 + 12r + 4 - 1 = 3(3r^2 + 4r + 1)$ is again a multiple of three. q.e.d.

5. (15 points) Consider the relation S from $\mathbb{R}$ to $\mathbb{R}$ given by xSy if $x - y \in \mathbb{Z}$.

   (a) Show that S is an equivalence relation.
   ANS: Reflexivity: If $x \in \mathbb{R}$ then $x - x = 0 \in \mathbb{Z}$ so xSx.
   Symmetry: If $x, y \in \mathbb{R}$ and xSy then $x - y \in \mathbb{R}$. Thus $y - x = -(x - y) \in \mathbb{R}$ so ySx.
   Transitivity: If $x, y, z \in \mathbb{R}$ and xSy and ySz then $x - y, y - z \in \mathbb{R}$. Hence $x - z = x - y + y - z \in \mathbb{R}$ so zSz. q.e.d.

   (b) Find three different real numbers a, b and c for which $\overline{a} = \overline{b} \neq \overline{c}$.
   ANS: $a = \frac{1}{2}$, $b = \frac{7}{2}$, and $c = 5$ work.

6. (20 points) If A is a set consider the relation
$R = \{((x, y), \{x, y\}) | (x \in A) \wedge (y \in A)\}$ from $A \times A$ to $\mathcal{P}A$.

   (a) Draw an arrow diagram (eg Fig 3.1.1.b) for R if $A = \{1, 2, 3\}$.
   ANS: The diagram has $3 \cdot 3 = 9$ dots on the left and $2^3 = 8$ on the right with 9 arrows.

   (b) Show that for any set A the relation R is a function.
   ANS: If $(x, y) \in A \times A$ then $S = \{x, y\}$ is the unique element of $\mathcal{P}(A)$ with $(x, y)RS$.

   (c) Show that for any set A the relation R is not onto.
   ANS: If $S \in \text{range}(R)$ then there are $x, y \in A$ with $S = \{x, y\}$ so $x \in S$ so S is not the empty set which is an element of $\mathcal{P}(A)$. q.e.d.

   (d) Show that for any set A wth at least two elements the relation R is not one-to-one.
   ANS: If A has at least two elements choose $x, y \in A$ with $x \neq y$ and hence $(x, y) \neq (y, x)$ but $R((x, y)) = \{x, y\} = \{y, x\} = R((y, x))$. q.e.d.