MAT 108 Homework 13 Solutions

Problems are from A Transition to Advanced Mathematics 8th edition by Smith, Eggen, and Andre.

Section 2.4 #4cj, 5g, 6b, 12ace

4. Use the PMI to prove the following for all natural numbers n.

(c) $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$

Solution: (base case) The statement $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$ is true for $n = 1$ because $2^1 = 2^2 - 2 = 2$. (induction step) Assume as the inductive hypothesis that $\sum_{i=1}^{k} 2^i = 2^{k+1} - 2$ for some $k \in \mathbb{N}$. We must show that $\sum_{i=1}^{k+1} 2^i = 2k + 2 - 2$. Since $\sum_{i=1}^{k+1} 2^i$ is a finite sum, we can write separate out the first k terms and write it as $\sum_{i=1}^{k} 2^i + 2^{k+1}$. By the inductive hypothesis, we then have

$$\sum_{i=1}^{k+1} 2^i = \sum_{i=1}^{k} 2^i + 2^{k+1} = (2^{k+1} - 2) + 2^{k+1}$$

Writing $2^{k+1} + 2^{k+1} = 2 \cdot 2^{k+1} = 2^{k+2}$, we have shown $\sum_{i=1}^{k+1} 2^i = 2^{k+2} - 2$. Hence, the statement is true for $k + 1$. Thus, by the PMI, the statement is true for all $n \in \mathbb{N}$.

(j) $\prod_{i=1}^{n} \left(1 - \frac{1}{i+1}\right) = \frac{1}{n+1}$

Solution: (base case) The statement $\prod_{i=1}^{n} \left(1 - \frac{1}{i+1}\right) = \frac{1}{n+1}$ is true for $n = 1$ because $1 - \frac{1}{2} = \frac{1}{1+1} = \frac{1}{2}$.
(induction step) Assume as the inductive hypothesis that $\prod_{i=1}^{k} \left(1 - \frac{1}{i+1}\right) = 2\frac{1}{k+1}$ for some $k \in \mathbb{N}$. We must show that $\prod_{i=1}^{k+1} \left(1 - \frac{1}{i+1}\right) = \frac{1}{k+2}$. Since multiplication of real numbers is associative, we can write $\prod_{i=1}^{k+1} \left(1 - \frac{1}{i+1}\right) = (1 - \frac{1}{k+2}) \prod_{i=1}^{k} \left(1 - \frac{1}{i+1}\right)$. By the inductive hypothesis, we then have

$$\prod_{i=1}^{k+1} \left(1 - \frac{1}{i+1}\right) = \left(1 - \frac{1}{k+2}\right) \prod_{i=1}^{k} \left(1 - \frac{1}{i+1}\right) = \left(1 - \frac{1}{k+2}\right) \left(\frac{1}{k+1}\right).$$

Distributing and simplifying yields

$$\left(1 - \frac{1}{k+2}\right) \left(\frac{1}{k+1}\right) = \left(\frac{1}{k+1} - \frac{1}{(k+1)(k+2)}\right) = \frac{k+2-1}{(k+1)(k+2)} = \frac{k+1}{(k+1)(k+2)} = \frac{1}{k+2}.$$

Hence, the statement is true for $k + 1$. Thus, by the PMI, the statement is true for all $n \in \mathbb{N}$.

5. Use the PMI to prove the following for all natural numbers

(g) 8 divides $9^n - 1$.

Solution: (base case) The statement 8 divides $9^n - 1$ is true for $n = 1$ because 8 divides $9^1 - 1 = 8$. (induction step) Assume as the inductive hypothesis that 8 divides $9^k - 1$ for some $k \in \mathbb{N}$. We must show that $8|9^{k+1} - 1$. We first rewrite $9^{k+1} - 1$ as $(9 \cdot 9^k - 9 + 9) - 1 = 9(9^k - 1) + 9 - 1 = 9(9^k - 1) + 8$. By the inductive hypothesis, $9^k - 1$ is divisible by 8, so we can write $9^k - 1 = 8m$ for some $m \in \mathbb{N}$. Therefore, $9^{k+1} - 1 = 9(8m) + 8 = 8(9m + 1)$. Since $9m + 1$ is an integer, we have shown that $9^{k+1} - 1$ is divisible by 8. Thus, the statement holds for $k + 1$ and by induction the statement is true for all $n \in \mathbb{N}$.

6. Use the Generalized PMI to prove the following. Then show that the equation or inequality is false for some natural number n.

(b) $2^n > n^2$ for all $n > 4$.

Solution: (base case) The statement $2^n > n^2$ is true for $n = 5$ because $2^5 = 32 > 25 = 5^2$. (induction step) Assume as the inductive hypothesis that $2^k > k^2$ for some $k \in \mathbb{N}$, $k > 4$. We must show that $2^{k+1} > (k+1)^2$. Rewriting 2^{k+1} as $2(2^k)$, the inductive hypothesis implies that $2^{k+1} = 2(2^k) > 2k^2$. Since $k > 4$, we know that $k^2 > 4k > 2k + 1$. Therefore,

$$2^{k+1} > 2k^2 = k^2 + k^2 > k^2 + 2k + 1 = (k+1)^2.$$

Hence, the statement holds for $k+1$. By the generalized PMI, the statement is true for all $n \in \mathbb{N}$, $n > 4$.

Let $n = 3$. Then the statement is false because $2^3 = 8 < 9 = 3^2$.

12. 'Grade' the following proofs:

(a) (see textbook for proof)
Solution: F. The claim is clearly false and the attempted proof does not have a proper induction step. One logical flaw in their argument is that the two different sets of n horses are assumed to be the same color as each other. Specifically, two horses of different colors provide a counterexample two the attempted reasoning in the inductive step.

(c) (see textbook for proof)
Solution: F. The claim is clearly false. The attempted proof starts with the inductive hypothesis and manipulates it to try to show the claim, rather than starting with the $P(n+1)$ statement and using the inductive hypothesis to show it's true.

(e) (see textbook for proof)
Solution: F. The claim is false. The base case is false, since $1^2 + 1$ is even. Notably, the inductive step is solid.