MAT 108 Homework 18 Solutions

Problems are from A Transition to Advanced Mathematics 8th edition by Smith, Eggen, and Andre.

Section 3.2 #6be
Section 3.3 #7a, 10a, 11, 15abc

6. For each of the following, prove that the relation is an equivalent relation. Then give information about the equivalence classes as specified.

(b) The relation R on $\mathbb{N}$ given by mRn iff m and n have the same digit in the tens places. Find an element of $\overline{106}$ that is less than 50; between 150 and 300; greater than 1,000. Find three such elements in the equivalence class $\overline{635}$.

Solution: To show R is an equivalence relation we must show R is (i) reflexive, (ii) symmetric, and (iii) transitive:

(i) Let $n \in \mathbb{N}$ be a natural number. Then n has the same digit in the tens place as n, so nRn and R is reflexive.

(ii) Let $m, n \in \mathbb{N}$ such that mRn. Then m and n have the same digit in their tens places, so nRm. Therefore, R is symmetric.

(iii) Let $m, n, r \in \mathbb{N}$ and suppose that we have mRn and nRr. Since mRn, we know that m and n have the same digit in the tens place. Denote this digit by $k \in \{0, 1, 2, \ldots 9\}$. Furthermore, since nRr, we know that r must also have k in the tens place. Therefore, mRr.

Thus, since R satisfies (i), (ii), and (iii), R is an equivalence relation.

The number 2 is in $\overline{106}$ because we can write 2 as 02 and both 02 and 106 share a 0 in the tens place. Similarly, the number $202 \in \overline{106}$ and the number 1000 is in the equivalence class of $\overline{106}$.

For $\overline{635}$, we have $35, 36, 37 \in \overline{635}$.

(e) The relation T on $\mathbb{R} \times \mathbb{R}$ given by $(x, y)T(a, b)$ iff $x^2 + y^2 = a^2 + b^2$. Describe the equivalence classes of $(1, 2)$; of $(4, 0)$

Solution: We show R is (i) reflexive, (ii) symmetric, and (iii) transitive:

(i) Let $(x, y) \in \mathbb{R} \times \mathbb{R}$. Then $x^2 + y^2 = x^2 + y^2$, so $(x, y)R(x, y)$ and R is reflexive.

(ii) Let $(x, y), (s, t) \in \mathbb{R} \times \mathbb{R}$ be such that $(x, y)R(s, t)$. Then $x^2 + y^2 = s^2 + t^2$, so $s^2 + t^2 = x^2 + y^2$. Therefore, $(s, t)R(x, y)$ and R is symmetric.

(iii) Let $(x, y), (s, t), (u, v) \in \mathbb{R} \times \mathbb{R}$ and suppose that we have $(x, y)R(s, t)$ and $(s, t)R(u, v)$. Since $(x, y)R(s, t)$, we know that $x^2 + y^2 = s^2 + t^2$. Furthermore, since $(s, t)R(u, v)$, $s^2 + t^2 = u^2 + v^2$. By transitivity of equality, we have $x^2 + y^2 = u^2 + v^2$. Therefore, $(x, y)R(u, v)$.

Thus, since R satisfies (i), (ii), and (iii), R is an equivalence relation.

The equivalence class of $(1, 2)$ is all pairs of real numbers (a, b) satisfying $a^2 + b^2 = 5$; the equivalence class of $(4, 0)$ is all pairs of real numbers (a, b) satisfying. $a^2 + b^2 = 16$.

7. Describe the equivalence relation on each of the following sets with the given partition.

(a) $\mathbb{N}, \{\{1, 2, \ldots, 9\}, \{10, 11, \ldots, 99\}, \{100, 101, \ldots, 999\}, \ldots\}$.

Solution: (answers may vary) We can define our equivalence relation R to be mRn iff m and n have their first nonzero digit in the same spot.

10. Complete the proof of Theorem 3.3.2: Suppose that $\mathcal{P}$ is a partition of A and suppose that xQy if there exists $C \in \mathcal{P}$ such that $x \in C$ and $y \in C$. Prove that

(a) Q is symmetric.

Solution: Let $x, y \in A$ such that xQy. Then, by the definition of Q, we must have that there exists some $C \in \mathcal{P}$ such that $x, y \in C$. Since $x, y \in C$, that also means that yQx. Thus, Q is symmetric.

11. Let R be a relation on a set A that is reflexive and symmetric but not transitive. Let $R(x) = \{y \in A : xRy\}$. (Note that $R(x)$ is the same as $\overline{x}$ except that R is not an equivalence relation in this exercies.) Does the set $\mathcal{A} = \{R(x) : x \in A\}$ always form a partition of A? Prove that your answer is correct.

Solution: No. As a counterexample, let $A = \{x, y, z\}$ and take the relation $R = \{(x, x), (y, y), (z, z), (x, y), (y, x), (y, z), ($ Then R is reflexive and symmetric, but not transitive because $(x, y), (y, z) \in R$, but $(x, z) \notin R$. The set $\mathcal{A} := \{R(x) : x \in A\}$ does not form a partition of A. Indeed, $y \in R(x)$ and $y \in R(z)$, but $R(x) = \{x, y, \} \neq R(z) = \{y, z\}$.

15. 'Grade' the following proofs:

(a) (see textbook for proof)

Solution: F. The claim is true, but the attempted proof tries to show the implication $P \wedge \sim Q \implies \sim R$ by assuming $P \wedge Q$ and showing that this implies R. This is not a valid proof by contrapositive or any other means.

(b) (see textbook for proof)

Solution: A. Valid proof by contradiction.

(c) (see textbook for proof)

Solution: F. The claim is false if we let $A = B$. Then, part (ii) of the proof is not correct as stated, because if we have $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, then they may have nonempty intersection but still have $X \neq Y$.