MAT 108 Homework 21 Solutions

Problems are from A Transition to Advanced Mathematics 8th edition by Smith, Eggen, and Andre.

Section 4.1 #14d, 15d
Section 4.2 #2de, 5b, 6, 13, 19bce

14. By naming an equivalence class in the domain that is assigned to two different values, prove that the following are not well-defined functions.

(d) $f : \mathbb{Z}_8 \to \mathbb{Z}_5$ given by $f(\overline{x}) = [x + 4]$

Solution: Choose $\overline{1}$. Then we need $f(\overline{1}) = f(\overline{9})$, but $[1 + 4] = [5]$ and $[9 + 4] = [13]$. So we get $f(\overline{1}) = 0 \bmod 5$ and $f(\overline{9}) = 3 \bmod 5$.

15. Prove that the following function is well defined:

(d) the function $f : \mathbb{Z}_{12} \to \mathbb{Z}_4$ given by $f(\overline{x}) = [2x + 1]$.

Solution: Let $f : \mathbb{Z}_{12} \to \mathbb{Z}_4$ be given by $f(\overline{x}) = [2x + 1]$ and consider the equivalence class $\overline{x} \in \mathbb{Z}_{12}$. We must show that $f(\overline{x}) = f(\overline{y})$ for any $y \in \overline{x}$. By definition, we have $x = y + 12k$ for some $k \in \mathbb{Z}$. Then

$$f(\overline{x}) = [2x + 1] = [2(y + 12k) + 1] = [2y + 24k + 1].$$

Rewriting $24k = 4(6k)$ and noting that $6k \in \mathbb{Z}$ tells us that $[2y + 24k + 1]$ is equivalent to $[2y + 1] = f(\overline{y})$ modulo 4. Thus, $f(\overline{x}) = f(\overline{y})$, so f is a well-defined function.

2. Find $f \circ g$ and $g \circ f$ for each pair of real functions f and g. Use the understood domains for f and g.

(d) $f(x) = \tan x$, $g(x) = \sin x$

Solution: $f \circ g = f(\sin x) = \tan \sin x$ with domain $\mathbb{R}$, equal to the domain of sin.
$g \circ f = f(\tan(x)) = \sin(\tan x)$ with domain $(-\pi/2, \pi/2)$, to the domain of $\tan x$.

(e) $f(x) = \{(t, r), (s, r), (k, l)\}$, $g(x) = \{(k, s), (t, s), (s, k)\}$.

Solution: $f \circ g = \{(k, r), (t, r), (s, l)\}$ with domain $\{k, t, s\}$.
$g \circ f = \emptyset$.

5. Let $\mathbb{Z}_8 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}\}$ and $\mathbb{Z}_4 = \{[0], [1], [2], [3]\}$. Define $f : \mathbb{Z}_8 \to \mathbb{Z}_4, g : \mathbb{Z}_4 \to \mathbb{Z}_8, h : \mathbb{Z}_8 \to \mathbb{Z}_8$, and $k : \mathbb{Z}_4 \to \mathbb{Z}_4$ as follows: $f(\overline{x}) = [x + 2], g([x]) = \overline{2x}, h(\overline{x}) = \overline{2x + 4}$, and $k([x]) = [2x + 2]$. By comparing images, verify the following equalities.

(b) $(g \circ f)(x) = h(x)$ for all $x \in \mathbb{Z}_8$.

Solution: Let f, g, h be defined as in the statement of the problem. To show $g \circ f = h$, we compare images:

($x = \overline{0}$) We have $(g \circ f)(\overline{0}) = g([2]) = \overline{4}$ and $h(\overline{0}) = \overline{0 + 4} = \overline{4}$.
($x = \overline{1}$) We have $(g \circ f)(\overline{1}) = g([3]) = \overline{6}$ and $h(\overline{1}) = \overline{2 + 4} = \overline{6}$.
($x = \overline{2}$) We have $(g \circ f)(\overline{2}) = g([0]) = \overline{0}$ and $h(\overline{2}) = \overline{4 + 4} = \overline{0}$.
($x = \overline{3}$) We have $(g \circ f)(\overline{3}) = g([1]) = \overline{2}$ and $h(\overline{3}) = \overline{6 + 4} = \overline{2}$.

($x = \overline{4}$) We have $(g \circ f)(\overline{4}) = g([2]) = \overline{4}$ and $h(\overline{4}) = \overline{8+4} = \overline{4}$.
($x = \overline{5}$) We have $(g \circ f)(\overline{5}) = g([3]) = \overline{6}$ and $h(\overline{5}) = \overline{10+4} = \overline{6}$.
($x = \overline{6}$) We have $(g \circ f)(\overline{6}) = g([0]) = \overline{0}$ and $h(\overline{0}) = \overline{12+4} = \overline{0}$.
($x = \overline{7}$) We have $(g \circ f)(\overline{7}) = g([1]) = \overline{2}$ and $h(\overline{7}) = \overline{14+4} = \overline{2}$.

Thus, $(g \circ f)(\overline{x}) = h(\overline{x})$ for all $x \in \mathbb{Z}_8$.

6. Prove the remaining part of Theorem 4.2.3: If $f : A \to B$, then $I_B \circ f = f$.

Solution: Ask in OHs or on Piazza for solution.

13. Let $h : A \to B$ and $g : C \to D$, and suppose that $E = A \cap C$. Prove that $h \cup g$ is a function from $A \cup C$ to $B \cup D$ if and only if $h|_E = g|_E$.

Solution: Let A, B, C, D, E, g and h be given as in the statement of the problem. Before proving anything, we should first define

$$h \cup g := \{((x, y) : (x, y) \in h \text{ or } (x, y) \in g\}.$$

($\implies$) Assume that, as defined, $h \cup g$ is a function from $A \cup C$ to $B \cup D$ and let $x \in E$. Since $x \in E = A \cap B$, x is in both the domain of A and the domain of B. So $(x, y) \in h$ and $(x, z) \in g$ for some $y \in C$, $z \in D$. Since we have assumed that $h \cup g$ is a function, it is well-defined and we must have $y = z \in C \cap D$. Since $x \in E$ was arbitrary, we must have $h(x) = g(x)$ for all $x \in E$. Thus, if $h \cup g$ is a well defined function, then $h|_E = g|_E$.

($\impliedby$) Now assume that $h|_E = g|_E$. We must show that $h \cup g$ with domain $A \cup B$ is well-defined. Consider some $x \in A \cup B$. If $x \in A - B$, then $(x, y) \in h$ and $(x, y) \in h \cup g$. Since $x \notin B$, if we have any other $(x, z) \in h \cup g$, we must have $(x, z) \in h$. h is well-defined, so $(x, y), (x, z) \in h$ implies $y = z$. The same argument holds exchanging $B - A$ for $A - B$ and g for h. Therefore, we need only consider the case where $x \in A \cap B = E$. Suppose we have $x \in E$ such that $(x, y), (x, z) \in h \cup g$ for some $y, z \in C \cup D$. Suppose without loss of generality that $y \in C$ and $z \in D$. Then, since $x \in E$, we also know that $(x, y) \in h|_E$ and $(x, z) \in g|E$. But $h|_E = g|_E$ by assumption and these two functions are well-defined, so we must have $y = z \in C \cap D$. Therefore, $h \cup g$ is well-defined.

Note: to be completely correct, we should also verify that the domain of $h \cup g$ is $A \cup B$. Ask in OHs or Piazza for details.

19. 'Grade' the following proofs:

(b) (see textbook for proof)
Solution: C. In general, it is not true that $f = f \circ f$. This is a hypothesis. In the last sentence, the proof uses transitivity of equality, not cancellation.

(c) (see textbook for proof)
Solution: F. The claim is false in general and $(f \circ g) \neq (g \circ f)$ in general.

(e) (see textbook for proof)
Solution: F. We have shown in problem 13 that this claim is false as stated since we need the two functions to agree on $A \cap C$..