Math 108 Spring 2020 Practice Midterm
To receive full credit you must show all of your work.

1. If P, Q and R are propositions then take S to be the proposition $S = (P \vee (\sim Q)) \wedge R$.

   (a) Write out the truth table for S.

   ANS:

P	T	T	T	T	F	F	F	F
Q	T	T	F	F	T	T	F	F
R	T	F	T	F	T	F	T	F
S	T	F	T	F	F	F	T	F

   (b) Find truth values for P, Q and R so that the truth value of S differs from that for R.
   ANS: Only P being F, Q being T and R being T works.

2. (Same as 1): If A, B and C are sets then take D to be the set $D = (A \cup B^c) \cap C$.

   (a) Sketch a Venn (circle) diagram for D.

   (b) Find an example of sets A, B and C for which D differs from C.
   ANS: A empty and $B = C = \{a\}$ works.

3. For each statement below decide which of the universes $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ it is true in.

   (a) $(\forall a)(\forall b)5a + 3b = 11$. **ANS:** none

   (b) $(\forall a)(\exists b)5a + 3b = 11$. **ANS:** only $\mathbb{R}$

   (c) $(\exists a)(\forall b)5a + 3b \neq 11$. **ANS:** (negation of b) both $\mathbb{N}$ and $\mathbb{Z}$

   (d) $(\exists! a)(\exists b)5a + 3b = 11$. **ANS:** only $\mathbb{N}$

   (e) $(\exists a)(\exists b)5a + 3b = 11$. **ANS:** all three

   (f) $(\exists a)(\exists b)6a + 3b = 11$. **ANS:** only $\mathbb{R}$

4. Prove that if a and b are integers then ab is even iff either a is even or b is even.
   ANS: Sketch: This has the form $(P \iff Q_1 \vee Q_2)$ which is equivalent to $(P \implies Q_1 \vee Q_2$ and $Q_1 \vee Q_2 \implies P)$ which is equivalent to $(Q_1 \vee Q_2 \implies P$ and $\sim Q_1 \wedge \sim Q_2 \implies \sim P)$.
   The first says that if 2 divides a or b then 2 divides ab which is true. The second says that if a and b are odd then ab is odd which is true since if $a = 2r+1$ and $b = 2n+1$ then $ab = 4rn+2r+2n+1 = 2(2rn+r+n)+1$.

5. Every even natural number is less than its square.

   (a) Rewrite this sentence using quantifiers and logic notation.
   ANS: In the universe of natural numbers $(\forall n)(n \text{ is even}) \implies (n < n^2)$.

(b) Prove that the sentence is true.
ANS: (This is only a sketch.) There are may ways to prove this. One would be induction taking $P(n)$ to be the proposition that $2n < (2n)^2$. The base case is then $P(1)$ which is $2 < 4$ whch is true and the induction step would be that if $P(n)$ holds then $2(n+1) = 2n+2 < (2n)^2 + 2 < (2n)^2 + 4n + 4 = (2(n+1))^2$ so $P(n+1)$ holds.

6. Either prove or find a counterexample to the following statement:

   If B is a set and $\mathbb{A} = \{A_\alpha | \alpha \in \Delta\}$ is an indexed family of sets then $B - (\cap_{\alpha\in\Delta} A_\alpha) = \cap_{\alpha\in\Delta} (B - A_\alpha)$.
   ANS: This is false. Consider the counterexample with $\Gamma = \{a, b\}$, $A_a = \{\}$ and $A_b = B = \{1\}$. In this case the left hand side is $\{1\}$ but the right hand side is $\{\}$.