Math 108 Midterm Exam Solutions May 4, 2020- 3:10-4:00

1. (10 pts: Logic)
   If P and Q are propositions consider the proposition
   $S = (P \wedge Q) \vee (\sim P \vee \sim Q)$.
   (a) Write the truth table for the proposition S.
   (b) Determine whether $P \implies S$ is true.

   ANS: For (a) all four entires are T so (b) is also true.

2. (8 pts: Contrapositive)
   Assume that n is a natural number.
   Prove by contraposition that if n is not prime then $n^2 \neq 49$.
   ANS: If $n^2 = 49$ then $n = 7$ and hence n is prime.

3. (8 pts: Quantifiers)
   Which two of the following four are true? Explain your answer.
   In the universe of real numbers.
   (a) $(\forall x)(\exists y > x)(y^2 - 2y$ is positive.)
   (b) $\sim [(\forall x)(\exists y > x)(y^2 - 2y$ is positive.)]
   (c) $(\exists x)(\forall y < x)(y^2 - 2y$ is negative.)
   (d) $\sim [(\exists x)(\forall y < x)(y^2 - 2y$ is negative.)]

   ANS: (a) is true (take y to be the larger of 3 and $x+1$). (d) is equivalent to (a) and true while (b) and (c) are equivalent to its negation and false.

4. (8 pts: Venn)
   Prove that if A, B and C are sets then
   $C \subseteq A \cap B$ if and only if $C \subseteq A$ and $C \subseteq B$.
   ANS: $[C \subseteq A \cap B]$
   iff $[(x \in C) \implies x \in (A \cap B)]$
   iff $[(x \in C) \implies (x \in A) \wedge (x \in B)]$
   iff $[[(x \in C) \implies (x \in A)] \wedge [(x \in C) \implies (x \in B)]]$
   iff $[C \subseteq A$ and $C \subseteq B]$.

5. (8 pts: Induction)
   Prove that $(\forall n \in \mathbb{N}) \left(\sum_{r=1}^{n}(2r) = n(n+1)\right)$.
   ANS: The base case with $n = 1$ is $2 = 1(2)$ which is true.
   For induction assume LHS(n)=RHS(n). Hence LHS(n+1)=LHS(n)+2(n+1)=RHS(n)+2($n+1$)=$n(n+1) + 2(n+1)$=$(n+2)(n+1)$=RHS($n+1$).
   Therefore the result holds by PMI.

6. (8 pts: Power Set)
   Find sets A and B for which $\overline{\overline{(\mathcal{P}A) \cap (\mathcal{P}B)}} = 2$.
   (Here $\mathcal{P}A$ is the power set or set of subsets of A).
   ANS: $A = B = \{1\}$.