Math 127B Practice Midterm I Spring 2025

You may use one sheet of notes.

1. (40 points: Derivative and Straddling)

   (a) Show that if a function f defined on all real numbers ($\mathbb{R}$) has $f'(0) = 0$ then:

   $$[\forall(\epsilon > 0)\ \exists(\delta_\epsilon > 0)\ \forall\,(x, y \in \mathbb{R} \text{ with } -\delta_\epsilon < \mathbf{x} < \mathbf{0} < \mathbf{y} < \delta_\epsilon)]$$

   $$|f(y) - f(x)| < \epsilon(y - x).$$

   (b) Find an example of a function f defined on $\mathbb{R}$ with $f'(0) = 0$ for which the following is false:

   $$[\forall(\epsilon > 0)\ \exists(\delta_\epsilon > 0)\ \forall\,(x, y \in \mathbb{R} \text{ with } \mathbf{0} < \mathbf{x} < \mathbf{y} < \delta_\epsilon)]$$

   $$|f(y) - f(x)| < \epsilon(y - x).$$

   You need not prove that it is false just give a brief explaination of why this nonstraddling property fails for your function.

2. (20 points: MVT and Inflection)
   Show that if $f(x)$ is a function defined on $(0, 4)$ and

   (a) $f(x)$ is twice differentiable,

   (b) $f(1) = 1$,

   (c) $f(2) = 2$ and

   (d) $f(3) = 3$

   then there is some $c \in (0, 4)$ with $f''(c) = 0$.

3. (20 points: Convergence and Derivatives)
Find integers a and b so that the sequence of functions

$$\left\{ f_n(x) = \frac{\sin(n^b x)}{n^a} \right\}$$

defined on all reals satisfies all four of the following:

(a) The sequence is uniformly Cauchy.

(b) The pointwise limit f of the sequence is differentiable.

(c) The limit $L = \lim_{n\to\infty} f_n'(0)$ exists.

(d) $f'(0) \neq L$.

(Note that the sequence $\{f_n'(x)\}$ can not converge pointwise.)

4. (20 points Series: Weierstrass)
Show that the sequence

$$\left\{ f_n(x) = \sum_{t=1}^{n} \frac{\sin(3^t x)}{6^t} \right\}$$

converges pointwise to a differentiable function.