Math 127B Practice Midterm I Spring 2025

Solutions

Problem: 1 40 points: Derivative and Straddling

(a) Show that if a function f defined on all real numbers ($\mathbb{R}$) has $f'(0) = 0$ then:

$$[\forall(\varepsilon > 0)\ \exists(\delta_\varepsilon > 0)\ \forall(x, y \in \mathbb{R} \text{ with } -\delta_\varepsilon < x < 0 < y < \delta_\varepsilon)]$$
$$|f(y) - f(x)| < \varepsilon(y - x)$$

(b) Find an example of a function f defined on $\mathbb{R}$ with $f'(0) = 0$ for which the following is false:

$$[\forall(\varepsilon > 0)\ \exists(\delta_\varepsilon > 0)\ \forall(x, y \in \mathbb{R} \text{ with } 0 < x < y < \delta_\varepsilon)]$$
$$|f(y) - f(x)| < \varepsilon(y - x)$$

You need not prove that it is false just give a brief explaination of why this nonstraddling property fails for your function.

(a) Unwinding the statement that $f'(0) = 0$ says, we have that for any $\varepsilon > 0$ you can find a $\delta > 0$ such that for any $x \in \mathbb{R}$ that satisfies $0 < |x| < \delta$:

$$\left|\frac{f(x) - f(0)}{x}\right| < \varepsilon$$

Or, after multiplying both sides by $|x|$ to make this look more like what we want to show:

$$|f(x) - f(0)| < \varepsilon|x|$$

Still with the same ε and δ from the derivative unwinding:

$$\begin{aligned}
|f(y) - f(x)| &= |f(y) - f(0) + f(0) - f(x)| \\
&\leq |f(y) - f(0)| + |f(x) - f(0)| \\
&< \varepsilon(|y| + |x|) \\
&< \varepsilon(y - x) && (***)
\end{aligned}$$

In going to the (***) line, we are now supposing $-\delta < x < 0 < y < \delta$ (ie, negative x and positive y) so the absolute value bars can be simplified. The $\varepsilon > 0$ was arbitrary and the δ came from the derivative limit (ie, set $\delta_\varepsilon = \delta$ to match the problem statement).

(b) Rearranging the property we want to find a counterexample for this part:

$$\frac{|f(y) - f(x)|}{y - x} < \varepsilon$$

which (in a sense) says that the numerator goes to 0 much faster than the denominator. An idea for crafting a counterexample is to find a function f that oscillates near 0 and take points x on the crests and points y on the troughs. The stereotypical example of such a function differentiable at 0 is (example 8.10 of Hunter):

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

By running through the definition of the derivative, it can easily be shown that $f'(0) = 0$. Now take a sequence $y_n \to 0^+$ along the crests of f, and similarly $x_n \to 0^+$ along the troughs of f. If you want to be precise (not needed here), the first crest of $\sin(x)$ occurs at $x = \pi/2$ and trough $x = 3\pi/2$, so all crests and troughs occur at $x = \pi/2 + 2n\pi$ and $x = 3\pi/2 + 2n\pi$. So take $y_n = 1/(\pi/2 + 2n\pi)$ and $x_n = 1/(3\pi/2 + 2n\pi)$. When this is the case:

$$\frac{|f(y_n) - f(x_n)|}{y_n - x_n} = \frac{y_n^2 + x_n^2}{y_n - x_n}$$

Now $x_n^2, y_n^2 = O(1/n^2)$ and $y_n - x_n = O(1/n^2)$ (each is $O(1/n)$, but forming like denominators to subtract makes this $O(1/n^2)$) So as $n \to \infty$, we expect this ratio to go to a constant. Doing this numerically using the following code, I find this constant to be 0.63661985.

```fortran
program main
    use iso_fortran_env, only: dp => real64, terminal => output_unit
    implicit none
    real(dp), parameter :: pi = 4 * atan(1.0_dp)
    real(dp)            :: y, x
    integer             :: n, nmax

    nmax = 1000

    do n = 1, nmax
        y = yn(n)
        x = xn(n)
        write(terminal, "(f17.8)") abs(f(y) - f(x)) / (y - x)
    enddo


    contains

        real(dp) function yn(n)
            integer, intent(in) :: n

            yn = 1.0_dp / (pi/2.0_dp + 2.0_dp*n*pi)

        endfunction yn

        real(dp) function xn(n)
            integer, intent(in) :: n

            xn = 1.0_dp / (3.0_dp*pi/2.0_dp + 2.0_dp*n*pi)
```

```fortran
        endfunction xn

        real(dp) function f(x)
            real(dp), intent(in) :: x

            f = (x**2) * sin(1.0_dp / x)

        endfunction f

endprogram main
```

Problem: 2. 20 points: MVT and Inflection

Show that if $f(x)$ is a function defined on $(0,4)$ and

(a) $f(x)$ is twice differentiable,

(b) $f(1) = 1$,

(c) $f(2) = 2$ and

(d) $f(3) = 3$

then there is some $c \in (0,4)$ with $f''(c) = 0$.

The mean value theorem applied to f twice gives numbers $a \in (1,2)$ and $b \in (2,3)$ such that:

$$\begin{aligned} f'(a) &= \frac{f(2) - f(1)}{2 - 1} \\ &= 1 \\ f'(b) &= \frac{f(3) - f(2)}{3 - 2} \\ &= 1 \end{aligned}$$

Again using the mean value theorem, but on f' gives a number $c \in (a,b) \subseteq (0,4)$ such that:

$$\begin{aligned} f''(c) &= \frac{f'(b) - f'(a)}{b - a} \\ &= 0 \end{aligned}$$

Problem: 3. 20 points: Convergence and Derivatives

Find integers a and b so that the sequence of functions

$$\left\{ f_n(x) = \frac{\sin(n^b x)}{n^a} \right\}$$

defined on all reals satisfies all four of the following:

(a) The sequence is uniformly Cauchy.

(b) The pointwise limit f of the sequence is differentiable.

(c) The limit $L = \lim_{n\to\infty} f_n'(0)$ exists.

(d) $f'(0) \neq L$.

(Note that the sequence $\{f_n'(x)\}$ can not converge pointwise.)

A systematic way of doing this problem is to run down each criteria of f_n and finding restrictions on a and b from each one, and choosing a and b to satisfy all of them. I will go top-down, finding such restrictions.

First uniformly Cauchy. This means that for any $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that when $m, n > N$ (and without loss of generality set $m \geq n$) we have that $|f_m(x) - f_n(x)| < \varepsilon$. This is certainly satisfied if:

$$\begin{aligned} |f_m(x) - f_n(x)| &= \left| \frac{\sin(m^b x)}{m^a} - \frac{\sin(n^b x)}{n^a} \right| \\ &\leq \frac{1}{m^a} + \frac{1}{n^a} \\ &\leq \frac{2}{n^a} \\ &< \varepsilon \end{aligned}$$

which is satisfied if $a > 0$.

If we are assuming that $a > 0$, then the pointwise limit of $f_n(x)$ as $n \to \infty$ is 0 (n^a dominates $\sin(n^b x)$), which is differentiable.

Taking a derivative:

$$f_n'(x) = n^{b-a} \cos(n^b x)$$

And so:

$$f_n'(0) = n^{b-a}$$

In order for this to exist, we need $b \leq a$ (otherwise it blows up).

Earlier we showed that as long as $a > 0$, then $f(x) = \lim_{n\to\infty} f_n(x) = 0$, so $f'(0) = 0$. We now want $\lim_{n\to\infty} f_n'(0) = n^{b-a} \neq f'(0) = 0$. Take $b = a$, $n^{b-a} = 1$ and these numbers are different.

We have concluded that we get all desired properties when $b = a$ and $a > 0$. So take $a = 1$ and $b = 1$ for instance.

Problem: 4. 20 points Series: Weierstrass

Show that the sequence

$$\left\{ f_n(x) = \sum_{t=1}^{n} \frac{\sin(3^t x)}{6^t} \right\}$$

converges pointwise to a differentiable function.

I will be using the following theorem from Hunter to solve this problem. For reference, I will copy it here.

Theorem 9.18. Suppose that (f_n) is a sequence of differentiable functions $f_n : (a, b) \to \mathbb{R}$ such that $f_n \to f$ pointwise and $f_n' \to g$ uniformly for some $f, g : (a, b) \to \mathbb{R}$. Then f is differentiable on (a, b) and $f' = g$.

Using this theorem in this problem is a problem on technicalities. Our sequence of functions f_n for this problem is a sequence of partial sums. That sequence itself is defined as a partial sum of another sequence of functions $g_t(x) = \sin(3^t x)/6^t$. First I will show that f_n converges pointwise to a function by showing that it converges uniformly to a function by the Weierstrass test. To use it, note that:

$$\begin{aligned} |g_t(x)| &= \left| \frac{\sin(3^t x)}{6^t} \right| \\ &\leq \frac{1}{6^t} \\ &=: M_t \end{aligned}$$

Now $\displaystyle\sum_{t=1}^{\infty} M_t$ converges since it is a geometric series, so we can conclude that f_n converges uniformly to some function f (so it also does pointwise to f).

Taking a derivative:

$$\begin{aligned} f_n'(x) &= \sum_{t=1}^{n} g_t'(x) \\ &= \sum_{t=1}^{n} \frac{\cos(3^t x)}{2^t} \end{aligned}$$

Now you might be scared that I passed a derivative through a sum, but in this case it is fine since each n is finite. Now to again use the Weierstrass test, with constants N_t defined by:

$$\begin{aligned} |g_t'(x)| &= \left| \frac{\cos(3^t x)}{2^t} \right| \\ &\leq \frac{1}{2^t} \\ &=: N_t \end{aligned}$$

Again $\sum_{t=1}^{\infty} N_t$ converges since it is a geometric series, so f_n' converges uniformly to some function g in which $f' = g$. This finishes the proof that f_n converges pointwise to some differentiable function.