— Exercises —

Exercise 7.1. Use the definition of the derivative to find the derivatives of the following functions. For each, let the domain be the positive real numbers.

(a) $f(x) = \sqrt{x+3}$ (b) $g(x) = \frac{1}{x}$ (c) $h(x) = 4x + \frac{1}{x^2}$

Exercise 7.2. Assume f and g are functions defined on (a, b), both of which are differentiable at a point $c \in (a, b)$. Also, let $\alpha, \beta \in \mathbb{R}$. Apply the definition of the derivative to the function $(\alpha f + \beta g)$ to prove that $(\alpha f + \beta g)'(c) = \alpha f'(c) + \beta g'(c)$.

Exercise 7.3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \lfloor x \rfloor$, and let C be the collection of points on which f is differentiable. Determine C and determine the function f' (which is a function from C to $\mathbb{R}$).

Exercise 7.4. Prove that there do not exist differentiable function f and g for which $f(0) = g(0) = 0$ and $x = f(x)g(x)$.

Exercise 7.5. Prove the *quotient rule* (Theorem 7.12).

Exercise 7.6. Is

$$f(x) = \begin{cases} \frac{1}{2}x & \text{if } x \in \mathbb{Q}, \\ x & \text{if } x \notin \mathbb{Q} \end{cases}$$

differentiable at 0?

Exercise 7.7. Use induction to prove that $\frac{d}{dx}x^n = nx^{n-1}$ for all $n \in \mathbb{N}$, which is the natural number case of the *power rule*.

Exercise 7.8. Let

$$f_a(x) = \begin{cases} x^a & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

(a) For which values of $a \in \mathbb{R}$ is f_a continuous at 0?

(b) For which values of $a \in \mathbb{R}$ is f_a differentiable at 0?

Exercise 7.9. Assume $f : \mathbb{R} \to \mathbb{R}$.

(a) Let $t \in (1, \infty)$. Prove that if $|f(x)| \leq |x|^t$ for all x, then f is differentiable at 0.

(b) Let $t \in (0, 1)$. Prove that if $|f(x)| \geq |x|^t$ for all x, and $f(0) = 0$, then f is not differentiable at 0.

(c) Give a pair of examples showing that if $|f(x)| = |x|$ for all x, then either conclusion is possible.

Exercise 7.10. Recall that a function is *even* if $f(-x) = f(x)$ for all x, and is *odd* if $f(-x) = -f(x)$ for all x. The below two properties are true. Give two proofs of each — one using the definition of the derivative, and one using a result from this chapter — and also draw a picture of each to model the property.

(a) If $f : \mathbb{R} \to \mathbb{R}$ is even and differentiable, then $f'(-x) = -f'(x)$.

(b) If $f : \mathbb{R} \to \mathbb{R}$ is odd and differentiable, then $f'(-x) = f'(x)$.

Exercise 7.11.

(a) Prove that if f is differentiable at some c for which $f(c) \neq 0$, then $|f|$ is also differentiable at c.

(b) Give an example showing that part (a) no longer holds if $f(c) = 0$.

Exercise 7.12. Suppose f is a polynomial of degree n, and $f \geq 0$ (that is, $f(x) \geq 0$ for all x). Prove that

$$f + f' + f'' + \cdots + f^{(n)} \geq 0.$$

You may use, without proof, that $f \geq 0$ implies n is even, and you may use standard properties of even-degree, positive polynomials (e.g. f has a minimum value and $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty$).

Exercise 7.13. Assume that I is an interval and $f : I \to \mathbb{R}$ is differentiable. Prove that if f' is bounded, then f is uniformly continuous.

Exercise 7.14. Give an example of an interval I and a differentiable function $f : I \to \mathbb{R}$ which is uniformly continuous and for which f' is unbounded.

Definition 7.28. A function f is called *Lipschitz continuous* if there exists some $C \in \mathbb{R}$ where

$$|f(x) - f(y)| \leq C \cdot |x - y|$$

for all x and y.

Exercise 7.15. Let $f : [a, b] \to \mathbb{R}$ be differentiable, and assume f' is continuous on $[a, b]$. Prove that f is Lipschitz continuous on $[a, b]$.

Exercise 7.16. Let I be an interval and $f : I \to \mathbb{R}$ be differentiable. Show that f is Lipschitz on I if and only if f' is bounded on I.

Exercise 7.17. Which is greater, e^π or π^e?

Make sure to give a *proof* that your answer is correct — don't just quote your calculator. Feel free to use facts you learned in calculus about $\ln(x)$... and perhaps consider the function $f(x) = \frac{\ln(x)}{x}$.

Exercise 7.18. In this exercise you will explore why each of the three main assumptions in Rolle's theorem is necessary. To that end, give an example of a function f which satisfies of each of the following conditions, and yet $f'(c) \neq 0$ for all $c \in (a, b)$.

(a) A function f which is continuous on $[a, b]$ and differentiable on (a, b) (but we don't assume $f(a) = f(b)$).

(b) A function f which is continuous on $[a, b]$ and $f(a) = f(b)$ (but we don't assume f is differentiable on (a, b)).

(c) A function f which is differentiable on (a, b) and $f(a) = f(b)$ (but we don't assume f is continuous on $[a, b]$).

Exercise 7.19. Use the derivative to find all values of a for which $|x-a| = (x-2)^2$.

Exercise 7.20. Suppose f and g are differentiable functions with $f(a) = g(a)$ and $f'(x) < g'(x)$ for all $x > a$. Prove that $f(b) < g(b)$ for any $b > a$.

Exercise 7.21. A *fixed point* of a function f is a value c for which $f(c) = c$. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable for all x, and $f'(x) \neq 1$ for all $x \in [0, 1]$, then f can have at most one fixed point in $[0, 1]$.

Exercise 7.22. Prove that if f is differentiable and $f'(x) \geq M$ for all $x \in [a, b]$, then $f(b) \geq f(a) + M(b - a)$.

Exercise 7.23.

(a) Suppose that $g : [0, 5] \to \mathbb{R}$ is differentiable, $g(0) = 0$, and $|g'(x)| \leq M$. Show for all $x \in [0, 5]$ that $|g(x)| \leq Mx$.

(b) Suppose that $h : [0, 5] \to \mathbb{R}$ is twice-differentiable (meaning that it's differentiable, and its derivative is differentiable), that $h'(0) = h(0) = 0$ and $|h''(x)| \leq M$. Show for all $x \in [0, 5]$ that $|h(x)| \leq Mx^2/2$.

(c) Can you give a geometric interpretation of the previous two parts?

Exercise 7.24. Assume that $f(0) = 0$ and $f'(x)$ is an increasing function. Prove that $g(x) = f(x)/x$ is an increasing function on $(0, \infty)$.

Exercise 7.25. Let f be a differentiable function on the interval $[0, 3]$ satisfying that $f(0) = 1$, $f(1) = 2$ and $f(3) = 2$.

(a) Show that there is some point $c \in (0, 3)$ such that $f(c) = c$.

(b) Show that there is some point $d \in [0, 3]$ with $f'(d) = \frac{1}{3}$.

(c) Show that there is some point $e \in [0, 3]$ with $f'(e) = \frac{1}{4}$.

Exercise 7.26. Assume $f : (a, b) \to \mathbb{R}$ is differentiable at some point $c \in (a, b)$. Prove that if $f'(c) \neq 0$ there exists some $\delta > 0$ such that $f(x) \neq f(c)$ for all $x \in (c - \delta, c + \delta)$.

Exercise 7.27. Prove that among all rectangles with a perimeter of p, the square has the greatest area.

Exercise 7.28. Show that the sum of a positive number and its reciprocal is at least 2.

Exercise 7.29. Prove the *Cauchy mean value theorem* (Theorem 7.26), which says this:

> If f and g are continuous on $[a, b]$ and differentiable on (a, b), then there is a number $c \in (a, b)$ such that
>
> $$[f(b) - f(a)] \cdot g'(c) = [g(b) - g(a)] \cdot f'(c).$$

(Note that the mean value theorem (Theorem 7.22) is a special case of this, in which $g(x) = x$.)

Exercise 7.30. Suppose f is a strictly monotone continuous function (guaranteeing f's inverse function, f^{-1}, exists) on (a, b). Prove that if $f'(x_0)$ exists and is non-zero, then f^{-1} is differentiable at $f(x_0)$ and

$$(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}.$$

Use this to produce the familiar formula for the derivative of the square-root function.

Exercise 7.31. Prove that Thomae's function is not differentiable at any point.

Exercise 7.32. Prove *Leibniz's rule*. That is, prove that if f and g have n^{th} order derivatives on (a, b) and $h = f \cdot g$, then for any $c \in (a, b)$,

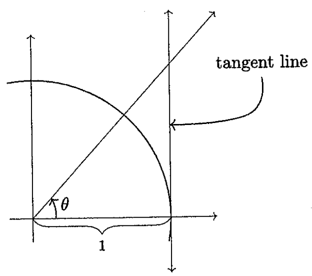
$$h^{(n)}(c) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(c) g^{(n-k)}(c).$$

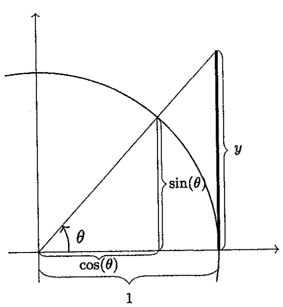
Exercise 7.33. Prove the following "∞/∞" case of l'Hôpital's rule. That is, prove that if $f : (a,b) \to \mathbb{R}$ and $g : (a,b) \to \mathbb{R}$ are differentiable functions and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$, then

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L \quad \text{implies} \quad \lim_{x\to a} \frac{f(x)}{g(x)} = L.$$

Exercise 7.34. In this exercise you will derive the formulas $\frac{d}{dx}\cos(x)$, $\frac{d}{dx}\sin(x)$ and $\frac{d}{dx}\tan(x)$.

First, do you know where $\tan(x)$ got its name? Consider a circle of radius 1, a vertical tangent line, and a ray emanating from the origin at an angle of θ from the positive x-axis.





We will focus on just a part of this tangent, and ask for the length of the line segment. Call this length y, and decorate this picture (using the definition of a sine and cosine).

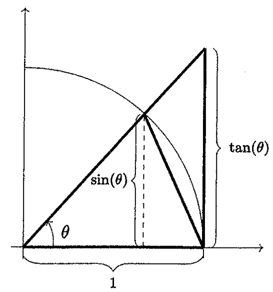
Do you see the similar triangles? The ratio of corresponding sides are thus equal:

$$\frac{\sin(\theta)}{\cos(\theta)} = \frac{y}{1}.$$

But $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$. So we have deduced that

$$y = \tan(\theta).$$

Remember where y came from: it was a tangent line. So now you see why the $\tan(\theta)$ function is call "tangent"—it's the length of a tangent! You will use this diagram to in this exercise:



(a) Notice that there is a triangle in this diagram whose base has length 1, and whose height has length $\sin(\theta)$. Notice that this triangle's area is less than the area of the portion of the circle in the θ angle, which is in turn less than the largest triangle's area. Use this reasoning to show that

$$\frac{1}{2}\sin(\theta) < \frac{1}{2}\theta < \frac{1}{2}\tan(\theta).$$

(b) Do some algebra to the above to show that

$$1 \geq \frac{\sin(\theta)}{\theta} \geq \cos(\theta).$$

(c) Use the above and the sequence squeeze theorem to prove that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

(d) Prove that

$$\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$$

by multiplying the numerator and denominator of this limit by $1+\cos(x)$ and, after some quick algebra, applying part (c).

Alternative definition of the derivative. The *derivative* of f is

$$\frac{d}{dx} f(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

(e) You will now use the above equivalent definition of the derivative to show that $\frac{d}{dx}\sin(x) = \cos(x)$. Here are the small steps:

(i) Write $\frac{d}{dx}\sin(x)$ by applying the equivalent definition.

(ii) Rewrite what you have by using the algebraic identity $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$.

(iii) Do some algebra so that you can apply the limits from parts (c) and (d).

(iv) Conclude that $\frac{d}{dx}\sin(x) = \cos(x)$.

(f) Now that you know $\frac{d}{dx}\sin(x) = \cos(x)$, use the algebraic identity $\cos(x) = \sin(\pi/2 - x)$ to show that $\frac{d}{dx}\cos(x) = -\sin(x)$.

(g) Now that you know $\frac{d}{dx}\sin(x) = \cos(x)$ and $\frac{d}{dx}\cos(x) = -\sin(x)$, show that $\frac{d}{dx}\tan(x) = \sec^2(x)$.

— Open Question —

Question. Notice that the formula for the area of a circle is πr^2. Differentiating this with respect for r gives $\frac{d}{dr}\pi r^2 = 2\pi r$, the formula for perimeter of the circle. The formula for the area of a square with side length $2r$ (so that its "radius" is r) is $(2r)(2r) = 4r^2$. Differentiating this with respect to r gives $\frac{d}{dr}4r^2 = 8r$, which is the perimeter of the square.

Classify all shapes where this property holds (including the higher-dimensional analogues, where you differentiate the volume formula to get the surface area formula). What other geometric interpretations of "radius" are there that give this property?