— Exercises —

Exercise 9.1. Let $f_k(x) = \dfrac{x\sin(x)}{k}$ on $[0, 10]$. Prove that (f_k) converges uniformly.

Exercise 9.2. Let $f_k(x) = x^k$ on $[0, 1]$, and let

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ 1 & \text{if } x = 1. \end{cases}$$

In Example 9.6 we argued that $f_k \to f$ pointwise. Prove that $f_k \not\to f$ uniformly.

Exercise 9.3. Let $f_k(x) = x/k$ on $\mathbb{R}$, and let $f(x) = 0$ on $\mathbb{R}$. Prove that (f_k) converges pointwise to f, but does not converge uniformly to f.

Exercise 9.4. For each of the following, determine the pointwise limit of (f_k) on the indicated interval, and state whether the convergence is uniform. You do not need thoroughly to prove your answers, but give a brief justification for your answer.

(a) $f_k(x) = \sqrt[k]{x}$, on $[0, 1]$.

(b) $f_k(x) = \dfrac{e^x}{k^2}$, on $(1, \infty)$.

(c) $f_k(x) = x^k - x^{2k}$ on $[0, 1]$.

(d) $f_k(x) = \dfrac{2kx}{1+k+3x}$ on $[0, \infty)$.

Exercise 9.5. Assume that, for each $k \in \mathbb{N}$, the function $f_k : A \to \mathbb{R}$ is unbounded.

(a) If $f_k \to f$ pointwise, must f be unbounded? Either prove that f must be unbounded or give a counterexample.

(b) If $f_k \to f$ uniformly, must f be unbounded? Either prove that f must be unbounded or give a counterexample.

Exercise 9.6. Assume that, for each $k \in \mathbb{N}$, the function $f_k : A \to \mathbb{R}$ is uniformly continuous. Also assume that $f_k \to f$ uniformly. Prove that f is uniformly continuous.

Exercise 9.7. Give an example of a sequence (f_k) of discontinuous functions, each defined on a domain I, for which $f_k \to f$ uniformly, for some continuous f.

Exercise 9.8. Complete Example 9.13 by showing that the f_k and f given in that example have the property that $f_k \to f$ uniformly.

Exercise 9.9. Give an example of a sequence (f_k) of integrable functions on $[a, b]$ for which $f_k \to f$ pointwise, but yet f is not integrable.

Exercise 9.10. Give an example of a sequence (f_k) of integrable functions on $[a, b]$ for which $f_k \to f$ pointwise, and f is integrable, but yet

$$\lim_{k\to\infty} \int_a^b f_k(x)\,dx \neq \int_a^b f(x)\,dx.$$

Exercise 9.11. Prove the *Cauchy Criterion for Uniform Convergence*: Let $f_k : A \to \mathbb{R}$. The sequence (f_k) converges uniformly if and only if for every $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $|f_m(x) - f_n(x)| < \varepsilon$ for all $m, n \geq N$ and all $x \in A$.

Exercise 9.12.

(a) Prove that if $f_k \to f$ uniformly on each of the sets $A_1, A_2, \ldots, A_n$, then f_k also converges uniformly to f on the set $\displaystyle\bigcup_{i=1}^{n} A_i$.

(b) Give an example to show that the above does not necessarily hold for countably many sets A_i.

Exercise 9.13. Suppose $f_k \to f$ pointwise and $g_k \to g$ pointwise, both on a domain $A \subseteq \mathbb{R}$.

(a) Prove that $(f_k + g_k) \to (f + g)$ pointwise on A.

(b) Does $(f_k \cdot g_k) \to (f \cdot g)$ pointwise? Prove or give a counterexample.

Exercise 9.14. Suppose $f_k \to f$ uniformly and $g_k \to g$ uniformly, both on a domain $A \subseteq \mathbb{R}$.

(a) Prove that $(f_k + g_k) \to (f + g)$ uniformly on A.

(b) Does $(f_k \cdot g_k) \to (f \cdot g)$ uniformly? Prove or give a counterexample.

Exercise 9.15. Give an example of a sequence (f_k) of functions where

(i) (f_k) converges uniformly on $[-10, 10]$, and

(ii) (f_k) converges pointwise on $\mathbb{R}$, but does not converge uniformly on $\mathbb{R}$.

Exercise 9.16. By Proposition 9.16, if $f_k \to f$ uniformly on $[a, b]$ and each f_k is integrable on $[a, b]$, then $\int_a^b f_k(x)\,dx \to \int_a^b f(x)\,dx$. We also showed in Example 9.6 and thereafter than $f_k(x) = x^k$ does *not* converge uniformly on $[0, 1]$. In this exercise, prove that uniform convergence is not a *necessary* condition for the convergence of a sequence of integrals by showing that if $f_k(x) = x^k$ on $[0, 1]$ and f is its pointwise limit, then

$$\int_0^1 f_k(x)\,dx \to \int f(x)\,dx$$

Exercise 9.17. Given an example of (1) a sequence of functions f_k on $[0,1]$, (2) a function f on $[0,1]$, and (3) a sequence (a_k) where each $a_k \in [0,1]$ and $a_k \to a$ for some a, such that the following conditions hold.

(i) Each f_k is continuous;

(ii) $f_k \to f$ pointwise; and

(iii) $f(a_k) \not\to f(a)$.

Exercise 9.18. Suppose (f_k) is a sequence of functions on $[a,b]$, each of which is monotone increasing. Suppose further that $f_k \to f$, for some f on $[a,b]$. Prove that f is monotone increasing on $[a,b]$.

Exercise 9.19. Give an example of a sequence (f_k) of functions on $[0,1]$, each of which is discontinuous at every point in $[0,1]$, but which converges uniformly to a continuous function.

Exercise 9.20. Let $f_k : A \to \mathbb{R}$. A sequence of functions (f_k) is said to be *uniformly bounded* if there exists some $C \in \mathbb{R}$ such that $|f_k(x)| \leq C$ for all $k \in \mathbb{N}$ and all $x \in A$.

(a) Explain why the property "each f_k is bounded" is different than the property "(f_k) is uniformly bounded."

(b) Give an example of a set A and a sequence (f_k) for which each f_k is bounded on A but the sequence (f_k) is not uniformly bounded.

Exercise 9.21. Prove the *Cauchy criterion for series of functions.* That is, let $f_k : A \to \mathbb{R}$, and prove that the series $\sum_{k=1}^{\infty} f_k$ converges uniformly on A if and only if for every $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that

$$\left|\sum_{k=m}^{n} f_k(x)\right| < \varepsilon$$

for all $n \geq m \geq N$ and all $x \in A$.

Exercise 9.22.

(a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, and define a sequence of functions by $f_n(x) = f(x + \frac{1}{n})$. Show that (f_n) converges uniformly to f.

(b) Give an example to show that this fails if we assume only that f is continuous and not uniformly continuous.

Exercise 9.23. Recall that $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x)$. Use this in the following.

(a) Show that, for $x \in (-1,1)$,

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots.$$

(b) Use part (a) to find a series expression for π.

Exercise 9.24. Show that the following series converge uniformly on the given interval.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(k^2 x)}{k^2}$ on $(-\infty, \infty)$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{x}{k^2}$ on $[-10, 10]$.

Exercise 9.25. Prove that if the series of functions $\sum_{k=1}^{\infty} f_k$ converges uniformly on a set A, then the sequence of functions (f_k) converges uniformly to 0 on A.

Exercise 9.26. Prove that the series $\displaystyle\sum_{k=1}^{\infty} \frac{k^2 + x^4}{k^4 + x^2}$ converges to a continuous function $f : \mathbb{R} \to \mathbb{R}$.

Exercise 9.27. Prove that the series

$$\sum_{k=1}^{\infty} 4^k \sin\left(\frac{1}{5^k x}\right)$$

converges uniformly on $[2, \infty)$. You may use the fact that $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

Exercise 9.28. Give an example of a series of functions $\sum_{k=1}^{\infty} f_k$ on some set A which converges uniformly, but for which the Weierstrass M-test fails.

Exercise 9.29. Determine whether the following converse to the Weierstrass M-test holds. Suppose that (f_k) is a sequence of nonnegative, bounded function on a set A, and let $M_k = \sup(\{f_k(x) : x \in A\})$. If $\sum_{k=1}^{\infty} f_k$ converges uniformly on A, does it follow that $\sum_{k=1}^{\infty} M_k$ converges?

Exercise 9.30. Prove that if a power series $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely at some $x_0 \in \mathbb{R}$, then it converges uniformly on the closed interval $\big[-|x_0|, |x_0|\big]$.

Exercise 9.31. Prove that if a power series $\sum_{k=0}^{\infty} a_k x^k$ has the property that $0 < m \leq |a_k| \leq M < \infty$ for some m and M, and for all k, then the power series has a radius of convergence of 1.

Exercise 9.32. Recall that the Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$ is defined by $a_1 = a_2 = 1$ and $a_k = a_{k-1} + a_{k-2}$.

(a) Show that for every $k \in \mathbb{N}$ the inequality $\frac{a_{k+1}}{a_k} \leq 2$ holds.

(b) Show that the power series

$$\sum_{k=1}^{\infty} a_k x^{k-1} = 1 + 1x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \cdots$$

converges for $|x| < 1/2$.

(c) Prove that for $|x| < 1/2$ the power series in the previous part converges to

$$f(x) = \frac{-1}{x^2 + x - 1}.$$

(d) Use partial fractions to obtain another power series for $f(x)$.

(e) The two power series must be the same. Conclude that

$$a_k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k.$$

(f) Reflect on the strange beauty of the formula from the previous part.

Exercise 9.33. Find the Taylor series for the following functions about the given point.

(a) $f(x) = \sin(x)$ about $x = 0$.

(b) $g(x) = \cos(2x)$ about $x = \pi/6$.

(c) $h(x) = x^3$ about $x = 2$.

(d) $s(x) = \frac{1+x}{1-x}$ about $x = 0$.

(e) $t(x) = \sqrt{x}$ about $x = 9$.

(f) $u(x) = \frac{1}{x}$ about $x = 4$.

Exercise 9.34. Use Theorem 9.31 and the Maclaurin series expansions for $\sin(x)$, $\cos(x)$ and e^x to show that

$$\frac{d}{dx}e^x = e^x \qquad \text{and} \qquad \frac{d}{dx}\sin(x) = \cos(x) \qquad \text{and} \qquad \frac{d}{dx}\cos(x) = -\sin(x).$$

Exercise 9.35. Use the Maclaurin series expansion for e^x to show $e^x e^y = e^{x+y}$.

Exercise 9.36. Let $f(x) = \ln|x+1|$.

(a) Find the Maclaurin series for f.

(b) State where this series converges.

(c) Use this to find a series representation for $\ln(2)$.

Exercise 9.37. In this exercise you will prove that e is irrational.

(a) Show that for any n,

$$e = e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} + E_n(1),$$

where $0 < E_n(1) < \frac{3}{(n+1)!}$.

(b) Assume for a contradiction that $e = a/b$, for some $a, b \in \mathbb{N}$. That is,

$$\frac{a}{b} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} + E_n(1).$$

Since this holds for all n, pick an n so that the rest of the proof works out.

(c) Multiply both sides of the above by $n!$. For your chosen n, argue that $n!E_n(1)$ must be an integer. Then explain why this is this a contradiction.

— Open Questions —

Question 1. Assume that $f(x) := \sum_{k=1}^{\infty} a_k x^k$ is convergent for all x. What is a necessary and sufficient condition on the sequence (a_k) for f to be a bounded function?[8]

Question 2. Let $R(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials with integer coefficients and $Q(0) \neq 0$. Is there an algorithm that, given $P(x)$ and $Q(x)$ as input, always halts and correctly decides if the Taylor series of $R(x)$ at $x = 0$ has a constant coefficient of 0?

[8] As two examples, note that if all but finitely many of the a_k are zero, then f is a non-constant polynomial which is certainly unbounded. However, if $a_k = \frac{(-1)^k}{k!}$ for odd k and $a_k = 0$ for even k, then $f(x) = \sin(x)$, which is bounded. As you can see, there has to be some intricate cancellation occurring to achieve boundedness.