Math 127B Winter 2021 Final Exam

Due Friday March 19 at Noon

Feel free to work on the exam for the full 48 hours that it is available and make use of the materials (texts, notes, lectures) from the course. Do not discuss the problems with others or make use of other assistance. Please write and sign the course honor code on the exam: I pledge on my honor that I have not given or received any unauthorized assistance on this examination.

1. (17 points)(Derivative)
   (a) Show that if $f : [-1, 1] \to \mathbb{R}$ with $|f(x)| \le |x|^4$ then $f'(0)$ exists.
   (b) Find a function $g : [-1, 1] \to \mathbb{R}$ with $|g(x)| \le |x|^4$ for which $g''(0)$ does not exist.

2. (16 points)(MVT) Consider a smooth function $r : \mathbb{R} \to \mathbb{R}$ for which $r(0) = r(1) - 2 = r(2) - 2 = r(3) = r(4)$.
   Show that $r''(x) = -1$ for some value x.

3. (17 points)(Uniform) Show that if (a_n) is an absolutely convergent series of real numbers then
$$\sum_{n=0}^{\infty} a_n e^{\frac{x}{2^n}}$$
   converges pointwise to a smooth function on $\mathbb{R}$.

4. (17 points)(Banach) Fix a partition R of $[0, 1]$.
   Show that $(C[0, 1], U(| \bullet |; R))$ is a normed linear space. That is:
   (a) Show that if $h \in C[0, 1]$ then $U(|h|; R) = 0$ if and only if h is the zero function.
   (b) Show that if $h \in C[0, 1]$ and $c \in \mathbb{R}$ then $U(|ch|; R) = |c|U(|h|; R)$.
   (c) Show that if $h, k \in C[0, 1]$ then $U(|h + k|; R) \le U(|h|; R) + U(|k|; R)$.

5. (17 points)(Power Series) Consider
$$S(x) = \int_{t=0}^{x} \frac{dt}{(1 - t^2)^2}.$$
   (a) Find the Taylor polynomial $P_5(x)$ for $S(x)$ about 0.
   (b) Find the radius of convergence for the Taylor series for $S(x)$ about 0.
   (c) Find the radius of convergence for the Taylor series for $S(x)$ about $\frac{1}{3}$.

6. (16 points)(Integral) Find two partitions P and Q of $[0, 1]$ with every interval of P longer than every interval of Q and two bounded functions $a, b : [0, 1] \to \mathbb{R}$ with $U(a; P) < U(a; Q)$ and $U(b; P) > U(b; Q)$.

7. (17 points Extra Credit)(p.v. Parts) Consider a principal value version of the integration by parts formula:
$$p.v. \int_{-1}^{1} u(t)v'(t)dt = -p.v. \int_{-1}^{1} u'(t)v(t)dt + [u(t)v(t)]|_{-1}^{1}.$$
   (a) Check that if $u(t) = \frac{1}{t}$ and $v(t) = e^t - 1$ the formula holds.
   (b) Check that if $u(t) = \frac{1}{t}$ and $v(t) = e^t$ the formula does not hold (as only one side converges).