Math 127B Spring 2020 Practice Final

1. Consider the functions $f_n(x) = \sum_{k=1}^{n} \frac{e^{-kx}}{\sqrt{k}}$.

   (a) Show that f_n converges pointwise in $(0, \infty)$ but not at 0 and call the limit $f(x)$.

   (b) Determine whether the convergence is uniform.

   (c) Show that f' exists in $(0, \infty)$.

   (d) Show that $\int_{x=0}^{\infty} f(x)dx$ exists.

2. Show that the vector space of real analytic functions f in $\mathbb{R}$ which are equal to their own tenth derivative ($f^{(10)} = f$) is (exactly) 10 dimensional.

3. Show that if $S \subset [0, 1]$ is countable then there is f which is Riemann integrable in $[0, 1]$ with $\int_{x=0}^{1} f(x)dx = 0$ and $(\forall s \in S)$ we have $f(s) > 0$.

4. Find bounded functions f and g on $[-1, 1]$ so that $(fg)'''(0)$ does not exist but for every bounded function k on $[-1, 1]$ we have $(fk)''(0)$ exists.

5. If $\{f_n\}$ is a sequence of functions converging pointwise to f define $T(\{f_n\}) = F$ with $F(x) = \lim_{n\to\infty} n(f_n(x) - f(x))$ if it exists.
   Show that if $T(\{f_n\})$ and $T(\{g_n\})$ exist then $T(\{f_n g_n\})$ exists.

6. Show that if f and g are differentiable in $\mathbb{R}$ and periodic with $f(x + 1) = f(x)$ and $g(x + 1) = g(x)$ and $|f'| + |g'| \geq 1$ then for every $r \in \mathbb{R}$ there is c with $\frac{f'(c)}{g'(c)} = r$.

7. Show that if $f : (0, 1] \to \mathbb{R}$ is monotone decreasing then

   $$\sum_{n=1}^{\infty} \frac{1}{n^2} f\left(\frac{1}{n}\right)$$

   converges iff the improper integral

   $$\int_{x=0}^{1} f(x)dx$$

   converges.

   Hint: Consider also the two series $\sum \frac{1}{n^2 \pm n} f(\frac{1}{n})$ and their difference.